PROBLEM SOLVING MASTERCLASS WEEK 5

1. Let a and n be integers and let p be a prime such that $p > |a| + 1$. Prove that the polynomial $f(x) = x^n + ax + p$ cannot be represented as a product of two nonconstant polynomials with integer coefficients. (Youngjun, Romanian Olympiad)

2. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2. (Sarah, Putnam 1996A5)

3. Consider a triangle S in 3-space, and a fixed plane π such that the triangle and the plane do not intersect. Assume the sun is direclty above the plane so that the triangle casts a shadow onto the plane (i.e. the shadow is an orthogonal projection of the triangle onto the plane). Call the image triangle S'. Show that S' always fits inside S. (Kiyoto)

4. Let A be a matrix in $SO(n)$, i.e. an $n \times n$ matrix with determinant 1, whose n column vectors are orthonormal. If $0 < k < n$, show that the determinant of the $k \times k$ matrix in the upper-left corner equals the determinant of the $(n-k) \times (n-k)$ matrix in the lower-right corner. (One interesting consequence: Show that the area of the shadow of a unit cube is equal to its "height", i.e. the difference in height between its highest and lowest points. Hence find the area of the largest possible shadow of a unit cube, and of the smallest.) (Vin)

5. Let $A(a, b, c)$ be the area of a triangle with sides a, b, c. Let $f(a, b, c) = \sqrt{A(a, b, c)}$. Prove that for any two triangles with sides a, b, c and a', b', c' we have

$$f(a, b, c) + f(a', b', c') \leq f(a + a', b + b', c + c').$$

Wen do we have equality? (Paul, Putnam 1982B6)

6. Let $P(t)$ be a nonconstant polynomial with real coefficients. Prove that the system of simultaneous equations

$$0 = \int_0^x P(t) \sin t \, dt = \int_0^x P(t) \cos t \, dt$$

has only finitely many real solutions x. (Alex, Putnam 1980A5)

7. Given an arbitrary triangle, find the circumscribed (inscribed) ellipse with the smallest (largest) area. (Yuanli)

8. Prove that there are unique positive integers a, n such that $a^{n+1} - (a + 1)^n = 2001$. (Frank, Putnam 2001A5)

Date: Monday, November 24, 2003.

9. Show that for every positive integer n,

$$\left(\frac{2n-1}{e}\right)^{\frac{2n-1}{2}} < 1 \cdot 3 \cdot 5 \cdots (2n-1) < \left(\frac{2n+1}{e}\right)^{\frac{2n+1}{2}}.$$

(Ravi, Putnam 1996B2)

Also, here is the one-sentence proof of the theorem that each prime congruent to 1 mod 4 is the sum of two squares. (I mentioned it a couple of weeks ago.) The proof is by Don Zagier.

The involution on the finite set $S = \{\, (x,y,z) \in \mathbb{Z}^{\geq 0} : x^2 + 4yz = p \,\}$ defined by

$$(x,y,z) \mapsto \begin{cases} (x+2z, z, y-x-z) & \text{if } x < y-z \\ (2y-x, y, x-y+z) & \text{if } y-z < x < 2y \\ (x-2y, x-y+z, y) & \text{if } x > 2y \end{cases}$$

has exactly one fixed point $(1, 1, (p-1)/4)$, so $\#S$ is odd and the involution defined by $(x,y,z) \mapsto (x,z,y)$ also has a fixed point.

This handout can be found at

http://math.stanford.edu/~vakil/putnam03/

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