PROBLEM SOLVING MASTERCLASS WEEK 4

1. (Repeat) A stick is broken at random in two places. What is the probability that the three pieces can form a triangle? (A 20-second solution by Andy)

2. (Repeat) A rectangle is tiled with smaller rectangles. Each smaller rectangle has a side of integer length. Show that the same is true of the larger rectangle. (Another 20-second solution, by Ravi)

3. Prove that there exists a *unique* function f from the set $\mathbb{R}^+$ of positive real numbers to $\mathbb{R}^+$ such that

$$f(f(x)) = 6x - f(x) \qquad \text{and} \qquad f(x) > 0 \qquad \text{for all } x > 0.$$

(Frank, Putnam 1988A5)

4. Prove that the decimal part of

$$\left(5 + \sqrt{26}\right)^n$$

begins with either n zeros or n nines for all positive integers n. (Andy)

5. Let S denote the set of rational numbers different from $\{-1, 0, 1\}$. Define $f : S \to S$ by $f(x) = x - 1/x$. Prove or disprove that

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset,$$

where $f^{(n)}$ denotes f composed with itself n times. (Frank, Putnam 2001B4)

6. Let p be an odd prime and let $\mathbb{F}_p$ denote (the field of) integers modulo p. How many elements are in the set

$$\{\, x^2 : x \in \mathbb{F}_p \,\} \cap \{\, y^2 + 1 : y \in \mathbb{F}_p \,\}?$$

(Ravi, Putnam 1991B5) Follow-up: using this and unique factorization property of Gaussian integers ($\mathbb{Z}[i] = \{a + bi : a, b \in \mathbb{Z}\}$), show that every prime congruent to 1 modulo 4 is a sum of two squares.

This handout can be found at

http://math.stanford.edu/~vakil/putnam03/

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Date: Monday, November 17, 2003.