PROBLEM SOLVING MASTERCLASS WEEK 3

1. Prove that for $n \geq 2$,

$$2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}n \equiv 2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}\Big\}n-1 \pmod{n}.$$

(Yuanli, Putnam 1997B5)

2a. Does there exist a function $g : \mathbb{Z}^+ \to \mathbb{Z}^+$ with the property that $g(g(n)) = n + 1987$ for all n? (Frank, IMO 1987)

2b. (Follow-up) Prove that $f(n) = 1 - n$ is the only integer-valued function defined on the integers that satisfies the following conditions:

(i) $f(f(n)) = n$, for all integers n;
(ii) $f(f(n+2)+2) = n$ for all integers n;
(iii) $f(0) = 1$.

(Putnam 1992A1)

3a. Given any 5 distinct points on the surface of a sphere, show that we can find a closed hemisphere which contains at least 4 of them. (Andy, Putnam 2002A2)

3b. (Follow-up) A stick is broken at random in two places. What is the probability that the three pieces can form a triangle?

4. A function f is defined on the positive integers by

$$f(1) = 1, \qquad f(3) = 3, \qquad f(2n) = f(n)$$

$$\begin{aligned} f(4n+1) &= 2f(2n+1) - f(n), \\ f(4n+3) &= 3f(2n+1) - 2f(n) \end{aligned}$$

for all positive integers n. Determine the number of positive integers n, less than or equal to 1988, for which $f(n) = n$. (Shrenik, IMO 1988)

5. A rectangle is tiled with smaller rectangles. Each smaller rectangle has a side of integer length. Show that the same is true of the larger rectangle. (Ravi)

This handout can be found at

http://math.stanford.edu/~vakil/putnam03/

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