PUTNAM PROBLEM SOLVING SEMINAR WEEK 4

The Rules. These are way too many problems to consider. Just pick a few problems in one of the sections and play around with them. You are not allowed to try a problem that you already know how to solve.

Generating functions.

1. Suppose $p(x) = (1+x+x^2)^{2001}$ is expanded out as a huge degree 4002 polynomial

$$p(x) = a_0 + a_1 x + \cdots + a_{4001}x^{4001} + a_{4002}x^{4002}.$$

(a) Find the sum of the coefficients of $p(x)$. (b) Find the sum of the even coefficients of $p(x)$. (Hint: What is $p(-1)$?) (A challenge: Find the sum of every third coefficient — it turns out to be a power of 3.)

2. Suppose

$$x = 0.12345... = \sum_{i=1}^{\infty} \frac{i}{10^i}.$$

(a) What is the thousandth digit of x after the decimal place? (b) Show that x is a rational number. Find it.

3. Show that

$$0.0001\,0016\,0081\,0256\,... = \sum_{i=1}^{\infty} \frac{i^4}{10^{4i}}$$

is a rational number. (Tip: Don't try to find it!)

4. $1/9899 = 0.00\,01\,01\,02\,03\,05\,08\,13\ldots.$ (As in the previous problem, the spaces were added to make the pattern clear.) Explain! Can you generalize it? For example, which rational number is $0.000\,001\,001\,002\ldots$?

5. Notice that $e^{ax}e^{bx} = e^{(a+b)x}$. Consider both sides as power series. Write down the coefficient of x^n on each side. What equality have you just proved? (Recall that $e^y = \sum_{k=0}^{\infty} y^k/k!$.)

6. For nonnegative integers n and k, define $Q(n,k)$ to be the coefficient of x^k in the expansion of $(1+x+x^2+x^3)^n$. Prove that

$$Q(n,k) = \sum_{j=0}^{k} \binom{n}{j}\binom{n}{k-2j},$$

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where $\binom{a}{b}$ is the standard binomial coefficient. (Reminder: For integers a and b with $a \geq 0$, $\binom{a}{b} = \frac{a!}{b!(a-b)!}$ for $0 \leq b \leq a$, with $\binom{a}{b} = 0$ otherwise.) Here's a hint which may help: Note that $(1 + x + x^2 + x^3)$ factors.

Analysis on the real line: Handy facts to know (for the Putnam, and more generally for a long and happy life).

Convergence. The Dominated Convergence Theorem. The Monotone Convergence Theorem. Limit Comparison Test. Integral Comparison Test.

Continuity. The Intermediate Value Theorem. The Extreme Value Theorem. (More generally, a continuous function on a compact set attains its sup and inf.)

Descartes' Rule of Signs: If $p(x) = a_1 x^{r_1} + a_2 x^{r_2} + \cdots + a_k x^{r_k}$ is a polynomial with $a_i \in \mathbb{R}^*$ and $r_1 > r_2 > \cdots > r_k$, then the number of positive real zeros of $p(x)$ counted with multiplicity is the number of sign changes in the sequence $a_1, a_2, \ldots, a_k$ minus a nonnegative even integer.

Big O and little O notation: $O(g(n))$ is a stand-in for a function $f(n)$ for which there exists a constant C such that $|f(n)| \leq C|g(n)|$ for all sufficiently large n. (This does not necessarily imply that $\lim_{n\to\infty} f(n)/g(n)$ exists.) Similarly "$f(t) = O(g(t))$ as $t \to 0$" means that there exists a constant C such that $|f(t)| \leq C|g(t)|$ for sufficiently small nonzero t. $o(g(n))$ is a stand-in for a function $f(n)$ such that $\lim_{n\to\infty} f(n)/g(n) = 0$. One can similarly define "$f(t) = o(g(t))$ as $t \to 0$".

Calculus. Riemann Sums: if a function is Riemann-integrable, e.g. if it is continuous on a closed finite interval, then the integral is the limit of the Riemann sums.

Rolle's Theorem: Let $[a, b]$ be a closed interval in $\mathbb{R}$. Let $f(t)$ be a function that is continuous on $[a, b]$ and differentiable on (a, b), and suppose that $f(a) = f(b)$. Then there exists $c \in (a, b)$ such that $f'(c) = 0$.

Inequalities of integrals: $f \leq g$ means $\int_a^b f \leq \int_a^b g$ if $a \leq b$.

Taylor's Formula with Remainder: if h has continuous nth derivatives, then for any $x > 0$ and integer $n > 0$, there exists $\theta_n \in [0, x]$ such that

$$h(x) = h(0) + h'(0)x + \cdots + h^{(n-1)}(0)x^{n-1}/(n-1)! + h^{(n)}(\theta_n)x^n/n!.$$

Mean Value Theorem for integrals: If f is continuous on $[a, b]$, then for some c in $[a, b]$ we have $\int_a^b f(x)dx = f(c)(b - a)$. For derivatives: If f is continuous on $[a, b]$ and has a derivative at each point of (a, b), then there is a point c of (a, b) for which $f(b) - f(a) = f'(c)(b - a)$.

Always good to know. Ordinary differential equations

Random other facts.

Rouché's Theorem: If f and g are analytic functions on an open set of $\mathbb{C}$ containing a closed disc, and if $|g(z)-f(z)|<|f(z)|$ everywhere on the boundary of the disc, then f and g have the same number of zeros inside the disc.

Euler-Maclaurin Summation Formula: for any fixed $k>0$,

$$\sum_{j=a}^{b} f(j) = \int_a^b f(t)\,dt + \frac{f(a)+f(b)}{2} + \sum_{i=1}^{k} \frac{B_{2i}}{(2i)!}\left(f^{(2i-1)}(b) - f^{(2i-1)}(a)\right) + R_k(a,b),$$

where the Bernoulli numbers B_{2i} are given by the power series

$$\frac{x}{e^x-1} = 1 - x/2 + \sum_{i=1}^{\infty} \frac{B_{2i}}{(2i)!}x^{2i},$$

and the error term $R_k(a,b)$ is given by

$$R_k(a,b) = \frac{-1}{(2k+2)!}\int_a^b B_{2k+2}(t-\lfloor t\rfloor) f^{(2k+2)}(t)\,dt.$$

Application 1: Sum of kth powers. Application 2: Stirling's approximation to $n!$,

$$n! \sim \sqrt{2\pi n}\left(\frac{n}{e}\right)^n,$$

where the tilde indicates that the ratio of the two sides tends to 1 as $n\to\infty$.

Problems I'll discuss.

7. Let f be an infinitely differentiable real-valued function defined on the real numbers. If

$$f\left(\frac{1}{n}\right) = \frac{n^2}{n^2+1}, \qquad n = 1,2,3,\ldots,$$

compute the values of the derivatives $f^{(k)}(0)$, $k=1,2,3,\ldots$.

8. For any pair (x,y) of real numbers, a sequence $(a_n(x,y))_{n\ge 0}$ is defined as follows:

$$\begin{aligned} a_0(x,y) &= x, \\ a_{n+1}(x,y) &= \frac{(a_n(x,y))^2+y^2}{2}, \qquad \text{for } n\ge 0. \end{aligned}$$

Find the area of the region $\{(x,y) | (a_n(x,y))_{n\ge 0} \quad \text{converges}\}$.

9. Let a and b be positive numbers. Find the largest number c, in terms of a and b, such that

$$a^x b^{1-x} \le a\frac{\sinh ux}{\sinh u} + b\frac{\sinh u(1-x)}{\sinh u}$$

for all u with $0<|u|\le c$ and for all x, $0<x<1$. (Note: $\sinh u = (e^u - e^{-u})/2$.)

Other problems.

10. Suppose that a sequence a_1, a_2, a_3, $\ldots$ satisfies $0<a_n\le a_{2n}+a_{2n+1}$ for all $n\ge 1$. Prove that the series $\sum_{n=1}^{\infty} a_n$ diverges.

11. *Cauchy's Lemma.* Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function such that $f(x+y) = f(x) + f(y)$. Show that $f(x) = cx$ for some $c \in \mathbb{R}$.

12. *A first approximation to Stirling's formula.* Prove that $e(n/e)^n < n! < en(n/e)^n$. (Hint: Use Riemann sums on $y = \ln x$.)

13. Is there an infinite sequence $a_0, a_1, a_2, \ldots$ of nonzero real numbers such that for $n = 1, 2, 3, \ldots$ the polynomial

$$p_n(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

has exactly n distinct real roots?

14. Let N be the positive integer with 1998 decimal digits, all of them 1; that is,

$$N = 1111 \cdots 11.$$

Find the thousandth digit after the decimal point of $\sqrt{N}$.

This handout, and other useful things, can (soon) be found at

http://math.stanford.edu/˜vakil/stanfordputnam.html