PUTNAM PROBLEM SOLVING SEMINAR WEEK 3

The Rules. You are not allowed to try a problem that you already know how to solve. These are way too many problems to consider. Just pick a few problems in one of the sections and play around with them.

The Hints. Try small cases. Do examples. Look for patterns. Draw pictures. Use lots of paper. Talk it over. Choose effective notation. Look for symmetry. Divide into cases. Work backward. Argue by contradiction. Consider extreme cases. Eat pizza. Modify the problem. Generalize. Don't give up after five minutes. Don't be afraid of a little algebra. Sleep on it if need be. And ask!! If the problem has a 2001 in it, what happens if you replace 2001 by 1, or 2, or 3? What's important about 2001 — is it that it is odd, or divisible by 3, etc.?

Linear recursions (aka recurrence relations) and linear recursive sequences.

1.

(a) The sequence $q_1, q_2, \ldots$ satisfies $q_n = 3q_{n-2} - 2q_{n-3}$, and $q_0 = 0$, $q_1 = 3$, $q_2 = 11$. Find a general formula for q_n.

(b) What is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$?

(c) The sequence $r_1, r_2, \ldots$ satisfies $r_n = (5/2)r_{n-1} - r_{n-2}$, and $r_1 = 2001$. Suppose the sequence converges to a finite real number. Find r_2.

(d) The sequence $G_0, G_1, G_2, \ldots$ consists of every other Fibonacci number. Show that there is a linear recursion (e.g. of the form $G_n = aG_{n-1} + bG_{n-2}$). (Follow-up: How about a sequence consisting of every *tenth* Fibonacci number. How do you know there's a recursion? With integer coefficients?)

(e) Use the theory of linear recursive sequences to find a formula for the sequence $s_0 = 1$, $s_1 = 2$, $s_n = s_{n-2}$. What do you observe? Now try a sequence with period four, such as $t_0 = 1$, $t_1 = 0$, $t_2 = 0$, $t_3 = 0$.

2. Two ping pong players, A and B, agree to play several games. The players are evenly matched; suppose, however, that whoever serves first has probability P of winning that game (this may be player A in one game, or player B in another). Suppose A serves first in the first game, but thereafter the loser serves first. Let P_n denote the probability that A wins the nth game. Show that $P_{n+1} = P_n(1 - P) + (1 - P_n)P$. (Follow-up: If P is neither 0 nor 1, you might expect that the limit of P_n is $1/2$. Why? Can you show this, for example using the theory of linear recursive sequences?)

Date: October 23, 2001.

3.

(a) Let $I_n = \int_0^{\pi/2} \sin^n x \, dx$. Find a recurrence relation for I_n.
(b) Show that

$$I_{2n} = \frac{1 \times 3 \times 5 \times \cdots \times (2n-1)}{2 \times 4 \times 6 \times \cdots \times (2n)} \cdot \frac{\pi}{2}.$$

(c) Show that

$$I_{2n+1} = \frac{2 \times 4 \times 6 \times \cdots \times (2n-2)}{1 \times 3 \times 5 \times \cdots \times (2n-1)}.$$

(Fun follow-up: Write these formulas in terms of factorials. Hint: Can you see why $1 \times 3 \times \cdots \times (2n-1) = (2n)!/(2^n n!)$? Then try plugging $n = 1/2$ into the formula you get for (b); what do you get for $(1/2)!$? What's that $\sqrt{\pi}$ doing there?!)

4. Let $T_0 = 2$, $T_1 = 3$, $T_2 = 6$, and for $n \geq 3$,

$$T_n = (n+4)T_{n-1} - 4nT_{n-2} + (4n-8)T_{n-3}.$$

The first few terms are

$$2, 3, 6, 14, 40, 152, 784, 5168, 40576, 363392.$$

Find, with proof, a formula for T_n of the form $T_n = A_n + B_n$, where (A_n) and (B_n) are well-known sequences. (Hint: use all tools at your disposal, including inspired guesswork. Play around with the numbers. What do you notice?)

5. A gambling graduate student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly n points at some time in a sequence of n tosses is $(2 + (-1/2)^n)/3$. (Hint: Let P_n denote the probability of scoring exactly n points at some time. Express P_n in terms of P_{n-1}, or in terms of P_{n-1} and P_{n-2}. Use this linear recursion to give an inductive proof. Even better hint, useful in many circumstances: you've been given the answer, so reverse-engineer the recursion, and then try to prove it.)

Fancy linear algebra.

6. Calculate

$$\det \begin{pmatrix} 1 & 1 & 1 & 1 \\ a & b & c & d \\ a^2 & b^2 & c^2 & d^2 \\ a^4 & b^4 & c^4 & d^4 \end{pmatrix}$$

7.

(a) Let E_n denote the determinant of the n-by-n matrix having -1's below the main diagonal (from upper left to lower right) and 1's on and above the main diagonal. Show that $E_1 = 1$ and $E_n = 2E_{n-1}$ for $n > 1$.
(b) Let D_n denote the determinant of the n-by-n matrix whose (i,j)th element (the element of the ith row and jth column) is the absolute value of the difference of i and j. Show that $D_n = (-1)^{n-1}(n-1)2^{n-2}$.

(c) Let F_n denote the determinant of the n-by-n matrix with a on the main diagonal, b on the superdiagonal (the diagonal immediately above the main diagonal — having $n-1$ entries), and c on the subdiagonal (the diagonal immediately below the main diagonal — having $n-1$ entries). Show that $F_n = aF_{n-1} - bcF_{n-2}$, $n > 2$. What happens when $a = b = 1$ and $c = -1$?

(d) Evaluate the n-by-n determinant A_n whose (i,j)th entry is $a^{|i-j|}$ by finding a recursive relationship between A_n and A_{n-1}.

8. A matrix (m_{ij}) is circulant if the entry m_{ij} depends only on $j-i$ modulo n. Find the eigenvectors of a circulant n by n matrix. (Hint: Try the case $n=2$, and make a guess!)

9. Let D_n denote the value of the $(n-1)\times(n-1)$ determinant

$$\begin{vmatrix} 3 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 4 & 1 & 1 & \cdots & 1 \\ 1 & 1 & 5 & 1 & \cdots & 1 \\ 1 & 1 & 1 & 6 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 & \cdots & n+1 \end{vmatrix}.$$

Is the set $\{D_n/n!\}_{n\geq 2}$ bounded?

10. A sequence of convex polygons $\{P_n\}$, $n \geq 0$, is defined inductively as follows. P_0 is an equilateral triangle with sides of length 1. Once P_n has been determined, its sides are trisected; the vertices of P_{n+1} are the *interior* trisection points of the sides of P_n. Thus P_{n+1} is obtained by cutting corners off P_n, and P_n has $3\cdot 2^n$ sides. (P_1 is a regular hexagon with sides of length $1/3$.) Express $\lim_{n\to\infty} \text{Area}(P_n)$ in the form $\sqrt{a}/b$, where a and b are positive integers.

General problems.

11. Let $\mathbf{A}$ and $\mathbf{B}$ be different $n\times n$ matrices with real entries. If $\mathbf{A}^3 = \mathbf{B}^3$ and $\mathbf{A}^2\mathbf{B} = \mathbf{B}^2\mathbf{A}$, can $\mathbf{A}^2 + \mathbf{B}^2$ be invertible?

12. Define a *selfish* set to be a set which has its own cardinality (number of elements) as an element. Find, with proof, the number of subsets of $\{1, 2, \ldots, n\}$ which are *minimal* selfish sets, that is, selfish sets none of whose proper subsets is selfish.

13. Let C_1 and C_2 be circles whose centers are 10 units apart, and whose radii are 1 and 3. Find, with proof, the locus of all points M for which there exists points X on C_1 and Y on C_2 such that M is the midpoint of the line segment XY.

14. If $\mathbf{A}$ and $\mathbf{B}$ are square matrices of the same size such that $\mathbf{ABAB} = \mathbf{0}$, does it follow that $\mathbf{BABA} = \mathbf{0}$?

15. If a linear transformation A on an n-dimensional vector space has $n+1$ eigenvectors such that any n of them are linearly independent, does it follow that A is a scalar multiple of the identity? Prove your answer.

This handout, and other useful things, can (soon) be found at

http://math.stanford.edu/~vakil/stanfordputnam.html