A natural smooth compactification of the space of elliptic curves in projective space via blowing up the space of stable maps

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The moduli space of stable maps $\overline{\mathcal{M}}_{g,k}(X,\beta)$ to a complex projective manifold X (where g is the genus, k is the number of marked points, and $\beta \in H_2(X,\mathbb{Z})$ is the image homology class) is the central tool and object of study in Gromov-Witten theory. The open subset corresponding to maps from smooth curves is denoted $\mathcal{M}_{g,k}(X,\beta)$.

The protean example is $\overline{\mathcal{M}}_{0,k}(\mathbb{P}^n,d)$. This space is wonderful in essentially all ways: it is irreducible, smooth, and contains $\mathcal{M}_{0,k}(\mathbb{P}^n,d)$ as a dense open subset. The boundary

$$\Delta := \overline{\mathcal{M}}_{0,k}(\mathbb{P}^n,d) \setminus \mathcal{M}_{0,k}(\mathbb{P}^n,d)$$

is normal crossings. The divisor theory is fully understood, and combinatorially tractable [4]. In some sense, this should be seen as the natural generalization of the space of complete conics compactifying the space of smooth conics.

It is natural to wonder if such a beautiful structure exists in higher genus. In arbitrary genus, however, there is no reasonable hope: $\mathcal{M}_g(\mathbb{P}^n,d)$ is badly behaved. (We emphasize that even the *interior* of the moduli space of stable maps is badly-behaved.) More precisely, $\mathcal{M}_g(\mathbb{P}^n,d)$ (as g, n, and d vary) is arbitrarily singular in a well-defined sense — it can have essentially any singularity, and can have components of various dimension meeting in various ways with various nonreduced structures [6]. In short, there is no reasonable hope of describing a desingularization, as this would in essence involve describing a resolution of singularities.

In genus one, however, the situation remains remarkably beautiful. Although $\overline{\mathcal{M}}_{1,k}(\mathbb{P}^n,d)$ in general has many components, it is straightforward to show that $\mathcal{M}_{1,k}(\mathbb{P}^n,d)$ is irreducible and smooth. Let $\overline{\mathcal{M}}^0_{1,k}(\mathbb{P}^n,d)$ be the closure of this open subset (the "main component" of the moduli space).

We will describe a natural desingularization of this main component

$$\widetilde{\mathcal{M}}_{1,k}(\mathbb{P}^n,d) \to \overline{\mathcal{M}}^0_{1,k}(\mathbb{P}^n,d).$$

(Details appear in [7]. In particular, it is proved there that this construction actually gives a desingularization.) This desingularization has several desirable properties.

- It leaves the interior $\mathcal{M}_{1,k}(\mathbb{P}^n,d)$ unchanged.
- The boundary $\widetilde{\mathcal{M}}_{1,k}(\mathbb{P}^n,d) \setminus \mathcal{M}_{1,k}(\mathbb{P}^n,d)$ is simple normal crossings, with an explicitly described normal bundle.
- The points of the boundary have explicit geometric interpretations.
- The desingularization can be interpreted as blowing up "the most singular locus", then "the next most singular locus", and so on, but with an unusual twist.

- The divisor theory is explicitly describable, and the intersection theory is tractable. (For example, one can compute the top intersection of divisors using [9].)
- The compactification is natural in the following senses.
  (i) The desingularization is equivariant: it behaves well with respect to the symmetries of $\mathbb{P}^n$. Hence we can apply Atiyah-Bott localization to this space — not just in theory, but in practice.
  (ii) It behaves well with respect to the inclusion $\mathbb{P}^m \hookrightarrow \mathbb{P}^n$.
  (iii) It behaves well with respect to the marked points (forgetful maps, ψ-classes, etc.).
  (iv) Consider the universal map $\pi : \mathcal{C} \to \mathbb{P}^n$ over $\overline{\mathcal{M}}_{g,k}(\mathbb{P}^n, d)$, where $\rho : \mathcal{C} \to \overline{\mathcal{M}}_{g,k}(\mathbb{P}^n, d)$ is the structure morphism. An important sheaf in Gromov-Witten theory is $\rho_* \pi^* \mathcal{O}_{\mathbb{P}^n}(a)$. When $g > 0$, this is not a vector bundle, which causes difficulty in theory and computation. However, in genus 1, "resolving $\overline{\mathcal{M}}^0_{1,k}(\mathbb{P}^n, d)$ also resolves this sheaf": when the sheaf is pulled back to the desingularization, it "becomes" a vector bundle. More precisely, it contains a natural vector bundle, and is isomorphic to it on the interior. This vector bundle is explicitly describable.

We find it interesting that such a natural naive approach as we will describe actually works, and yields a desingularization with these nice properties. For example, if $n > 2$, this desingularization can be interpreted as a natural compactification of the Hilbert scheme of smooth degree d curves in projective space, and thus could be seen as the genus 1 version of the complete conics.

This construction also has a number of applications:

- enumerative geometry of genus 1 curves via localization.
- Gromov-Witten invariants in terms of enumerative invariants [8].
- the Lefschetz hyperplane property: effective computation of Gromov-Witten invariants of complete intersections [3] (see also [2] for the special case of the quintic threefold).
- algebraic version of "reduced" Gromov-Witten invariants in symplectic geometry [8].
- an approach to hopefully prove physicists' predictions [1] about genus 1 Gromov-Witten invariants (work of Zinger, in progress).

We finally describe the construction explicitly. (In the lecture, the geography of $\overline{\mathcal{M}}_{1,k}(\mathbb{P}^n, d)$ was sketched as motivation.) It is straightforward to show that $\overline{\mathcal{M}}_{1,k}(\mathbb{P}^n, d)$ is nonsingular on the locus where there is no contracted genus 1 (possibly nodal) curve (for example, the proof of [5, Prop. 4.21] applies). We say a stable map is in the *m-tail locus* if there is an arithmetic genus 1 contracted curve, with precisely m points of the contracted curve that are either marked, or meet the rest of the curve. The algorithm is then as follows: blow up the one-tail component (which actually does nothing to $\overline{\mathcal{M}}^0_{1,k}(\mathbb{P}^n, d)$), then the proper transform of all two-tail components, then the proper transform of all three-tail components, etc.

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