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On the tautological ring of $\overline{\mathcal{M}}_{g,n}$

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1. Introduction

The purpose of this note is to prove:

Theorem 1.1. $R_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$.

In other words, any two top intersections in the tautological ring are commensurate.

This provides the first genus-free evidence for an audacious conjecture of Faber and Pandharipande that the tautological ring is Gorenstein, answering in the affirmative a question of Hain and Looijenga; see Section 1.2.

1.1. Background on the tautological ring

In this section, we briefly describe the objects under consideration for the sake of non-experts. A more detailed informal exposition of these well-known ideas is given in [PV].

When studying Riemann surfaces of some given genus g, one is naturally led to study the moduli space $\mathcal{M}_g$ of such objects. This space has dimension $3g-3$, and has a natural compactification due to Deligne and Mumford, [DM], the moduli space $\overline{\mathcal{M}}_g$ of *stable* genus g curves. More generally, one can define a moduli space of stable n-pointed genus g curves, denoted $\overline{\mathcal{M}}_{g,n}$, over any given algebraically closed field (or indeed over $\operatorname{Spec}\mathbb{Z}$). We shall work over the complex numbers. A stable n-pointed genus g complex curve is a compact curve with only nodes as singularities, with n distinct labeled smooth points. There is a stability condition: each rational component has at least 3 special points, and each component of genus 1 has at least 1 special point. (A special point is a point on the normalization of the component that is either a marked point, or a branch of a node.) This stability condition is equivalent to requiring that the automorphism group of the pointed curve be finite. If a curve is stable, then a short combinatorial exercise shows that $2g-2+n>0$.

The open subset of $\overline{\mathcal{M}}_{g,n}$ corresponding to smooth curves is denoted $\mathcal{M}_{g,n}$. The curves of *compact type* (those with compact Jacobian, or equivalently, with a tree as dual graph) form a partial compactification, denoted $\mathcal{M}^c_{g,n}$.

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Even if one is interested primarily in nonsingular curves, i.e. $\mathcal{M}_g$, one is naturally led to consider the compactification, $\overline{\mathcal{M}}_g$; even if one is interested primarily in unpointed curves, one is naturally led to consider the space $\overline{\mathcal{M}}_{g,n}$ by the behavior of the boundary, $\overline{\mathcal{M}}_g \setminus \mathcal{M}_g$.

The space $\overline{\mathcal{M}}_{g,n}$ is best considered as an orbifold or Deligne-Mumford stack; it is nonsingular and proper (i.e. compact) of dimension $3g-3+n$, and hence has a good intersection theory. Statements about $\overline{\mathcal{M}}_{g,n}$ translate to universal statements about families of curves.

The cohomology of the moduli space is very interesting. For example, the cohomology of $\mathcal{M}_g$ is the group cohomology of the mapping class group. There is even further structure in the Chow ring of $\overline{\mathcal{M}}_{g,n}$ (an algebraic version of the cohomology ring).

The tautological ring, denoted $R^*(\overline{\mathcal{M}}_{g,n})$, is a subring of the Chow ring $A^*(\overline{\mathcal{M}}_{g,n})$. Natural algebraic constructions typically yield Chow classes lying in the tautological ring. A simple description of the tautological ring, due to Faber and Pandharipande ([FP] Section 0.3), is the following.

Define the cotangent line class $\psi_i \in A^*(\overline{\mathcal{M}}_{g,n})$ $(1 \leq i \leq n)$ as the first Chern class of the line bundle with fiber $T^*_{p_i}(C)$ over the moduli point $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$. The tautological system of rings $\{R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n})\}$ is defined as the set of smallest $\mathbb{Q}$-subalgebras satisfying the following three properties:

(i) $R^*(\overline{\mathcal{M}}_{g,n})$ contains the cotangent lines $\psi_1, \ldots, \psi_n$.

(ii) The system is closed under pushforward via all maps forgetting markings:

$$\pi_* : A^*(\overline{\mathcal{M}}_{g,n}) \to A^*(\overline{\mathcal{M}}_{g,n-1}).$$

(iii) The system is closed under pushforward via all gluing maps:

$$A^*(\overline{\mathcal{M}}_{g,n_1 \cup \{*\}}) \otimes_{\mathbb{Q}} A^*(\overline{\mathcal{M}}_{g_2,n_2 \cup \{\bullet\}}) \to A^*(\overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}),$$

$$A^*(\overline{\mathcal{M}}_{g,n \cup \{*,\bullet\}}) \to A^*(\overline{\mathcal{M}}_{g+1,n}).$$

The Hodge bundle $\mathbb{E}$ is the rank g vector bundle with fiber $H^0(C, \omega_C)$ over the moduli point $[C, p_1, \ldots, p_n] \in \overline{\mathcal{M}}_{g,n}$. The λ-classes are defined by $\lambda_k = c_k(\mathbb{E})$ $(0 \leq k \leq g)$; these classes (and many others) also lie in the tautological ring.

1.2. Is the tautological ring Gorenstein?

The study of the tautological ring was initiated in Mumford's foundational paper [M]. Recent interest in the tautological ring was sparked by Kontsevich's proof of Witten's conjectures, [K], which led to Faber's algorithm for computing all top intersections in the tautological ring, [F2].

In the early 1990's, Faber conjectured that $R^*(\mathcal{M}_g)$ is of a very special form (appearing in print much later, [F1]); it has the properties of the (p,p)-cohomology of a complex projective manifold of dimension $g-2$. In particular, he conjectured that it is a Gorenstein ring, i.e. $R^{g-2}(\mathcal{M}_g) = \mathbb{Q}$, and the intersection pairing

$$R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g)$$

is perfect (for $0 \leq i \leq g-2$). (Faber also gives a very specific description of top intersections, which determines the structure of the ring.) Key evidence for Faber's calculation was Looijenga's beautiful proof [L] that $R^n(\mathcal{M}_g)$ has dimension 0 for $n > g-2$ and at most 1 for $n = g-2$ (generated by the class of hyperelliptic curves), and Faber and Pandharipande's subsequent argument that $\dim R^{g-2}(\mathcal{M}_g) \geq 1$ (by explicitly showing that the class of hyperelliptic curves is non-zero, [FP] equ. (8)).

Motivated by Faber's conjecture, Hain and Looijenga asked if the tautological ring of $\overline{\mathcal{M}}_{g,n}$ satisfies Poincare duality ([HL] Question 5.5). The first evidence would be to check if Theorem 1.1 were true; they give an argument ([HL] Section 5.1) which Faber and Pandharipande note is incomplete ([FP] Section 0.6).

More generally, Faber and Pandharipande conjecture (or speculate) that:

1. $R^*(\mathcal{M}_g)$ is a Gorenstein ring with socle in codimension $g-2$,
2. $R^*(\mathcal{M}_g^c)$ is a Gorenstein ring with socle in codimension $2g-3$,
3. $R^*(\overline{\mathcal{M}}_g)$ is a Gorenstein ring with socle in codimension $3g-3$,

where R^* is the tautological ring of the appropriate space, and $\mathcal{M}_g^c$ is the moduli space of curves of compact type. There seems no reason (or counterexample) to prevent extensions of these conjectures to moduli spaces of pointed curves, and indeed such moduli spaces naturally come up inductively from the geometry of the boundary strata of the moduli spaces of unpointed curves.

The key first step in each case is to check that the part of the tautological ring in the highest (expected) codimension is one-dimensional, and this note addresses the third case (further allowing marked points). A proof of the first step in Cases 2 and 3 was known earlier to Faber and Pandharipande (manuscript in preparation). However, we feel this argument (for Case 3) is worth knowing, as it is short and conceptually simple.

Remark 1.1. There is no reason to expect $A_0\overline{\mathcal{M}}_{g,n} = \mathbb{Q}$ in general, and in fact there is reason to expect otherwise. For example, by a theorem of Srinivas ([S], based on earlier work of Roitman and Mumford), any normal projective variety X with a nonzero q-form ($q > 1$) defined on a nonsingular open subset missing a subset of codimension at least two has huge 0-dimensional Chow group. More precisely, the degree 0 dimension 0 cycle classes cannot be parametrized by an algebraic variety; in fact the image of the natural map from X^n to the degree n zero-cycles modulo rational equivalence is not surjective for any n. The coarse moduli scheme $\overline{M}_{g,n}$ and the fine moduli stack $\overline{\mathcal{M}}_{g,n}$ have the same Chow group (with $\mathbb{Q}$-coefficients, [Vi] p. 614), and $\overline{M}_{g,n}$ is a normal projective variety. For $n > 1$, the spaces $\overline{M}_{1,n}$ and $\overline{\mathcal{M}}_{1,n}$ are isomorphic away from the closed subset corresponding to pointed curves with nontrivial automorphisms; this set consists of the divisor Δ_1 on $\overline{M}_{1,n}$ (generically corresponding to a rational curve with n marked points, attached to a genus 1 curve with no marked points) and a set of codimension 2. Call the corresponding divisor on the fine moduli stack δ_1. It is not hard to show that if $n = 11$, and hence $n \geq 11$, $\overline{\mathcal{M}}_{1,n}$ has a nonzero 11-form vanishing on δ_1; this descends to a nonzero 11-form on the open subset of $\overline{M}_{1,n}$ corresponding to automorphism-free curves, which extends over (the general point of) Δ_1. Hence $\overline{M}_{1,n}$ satisfies the hypotheses of

Srinivas' theorem, and has huge Chow group. It does not seem unreasonable to suspect that for fixed $g > 0$ and sufficiently large n, $\overline{\mathcal{M}}_{g,n}$ will have a canonical form which descends to $\overline{M}_{g,n}$ (away from a subset of codimension two).

2. The ELSV formula in Chow

The essential ingredient is the remarkable formula of Ekedahl-Lando-Shapiro-Vainshtein ([ELSV1] Theorem 1.1, proved in [GV] and [ELSV2]):

Theorem 2.1. *Suppose g, n are integers ($g \geq 0$, $n \geq 1$) such that $2g-2+n > 0$, and $\alpha_1, \dots, \alpha_n$ are positive integers. Let H^g_α is the number of degree $\sum \alpha_i$ genus g irreducible branched covers of $\mathbb{P}^1$ with simple branching above r fixed points, branching with monodromy type α above ∞, and no other branching. Then*

$$H^g_\alpha = \frac{r!}{\# \operatorname{Aut}(\alpha)} \prod_{i=1}^n \frac{{\alpha_i}^{\alpha_i}}{\alpha_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{1-\lambda_1+\cdots\pm\lambda_g}{\prod(1-\alpha_i\psi_i)}.$$

We note that Theorem 2.1 actually holds in the Chow ring. Precisely, define the *Hurwitz class* $\mathbb{H}^g_\alpha$ to be the zero-cycle that is the sum of the points in $\overline{\mathcal{M}}_{g,n}$ corresponding to the source curves of the H^g_α covers, where the n marked points are the preimages of ∞. (Here, α should now be considered an n-tuple rather than a partition; the points above ∞ are now labeled.) This class is well-defined as all choices of the r branch points in $\mathbb{P}^1$ are rationally equivalent (as elements of $\operatorname{Sym}^r \mathbb{P}^1$).

Proposition 2.2. *In $A^*(\overline{\mathcal{M}}_{g,n})$,*

$$\frac{1}{r!} \prod_{i=1}^n \frac{\alpha_i!}{{\alpha_i}^{\alpha_i}} \mathbb{H}^g_\alpha = \left[\frac{1-\lambda_1+\cdots\pm\lambda_g}{\prod(1-\alpha_i\psi_i)} \right]_0 \tag{1}$$

Sketch of proof. Our proof of Theorem 2.1 in [GV] immediately shows that these classes are equal after pushing forward to $\overline{\mathcal{M}}_g$ under the map which forgets the marked points. To get the equality before the pushforward, a few minor modifications to the argument are needed. To obtain a moduli morphism to $\overline{\mathcal{M}}_{g,n}$, virtual localization must be used on a slightly different space. (Warning: in [GV], n is replaced by m.) We wish to count covers with n *marked* points over ∞, so we work on $\overline{\mathcal{M}}_{g,n}(\mathbb{P}^1, d)$ rather than $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$; this will change the formula by $\# \operatorname{Aut}(\alpha)$.

In [GV], M^k corresponds to the locus where the branching over ∞ is (at least) $\sum(\alpha_i - 1)$; it should be replaced by the locus where the branching over ∞ is (at least) $\sum(\alpha_i-1)$ *and* the n marked points map to ∞. In [GV], M^α corresponds to an irreducible component of M^k where the branching is of type α; it should be replaced by the irreducible component of M^k where the branching is of type α *and* where the point p_i is the branch point of order α_i. This gives an additional multiplicity of $\prod \alpha_i$ in m_α, which is cancelled by the same additional multiplicity in the virtual localization formula (coming from the presence of the marked points).

Remark 2.1. The proof of [ELSV2] should also be easily adapted to show Proposition 2.2, but we have not checked the details.

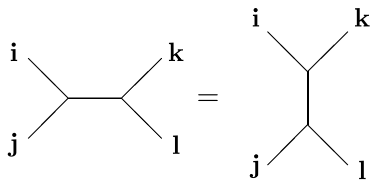

FIGURE 1. "Duality" equivalence relation for trivalent graphs

2.1.

Notice that the right side of (1) is a symmetric polynomial $P_{g,n}$ in the variables $\alpha_1, \dots,$ α_n (with degrees between $2g-3+n$ and $3g-3+n$), and the coefficients are monomials in ψ-classes and up to one λ-class. As the coefficients of such a polynomial can be expressed as linear combinations of the values of the polynomial evaluated at various n-tuples of positive integers, all monomials of ψ-classes are $\mathbb{Q}$-linear combinations of Hurwitz classes.

3. Proof of Theorem 1.1

The proof of the theorem is now a series of reduction statements.

3.1.

The smallest boundary strata (hereafter *top strata*) in $\overline{\mathcal{M}}_{g,n}$ correspond to curves whose dual graphs are trivalent (and all components are rational). It is well-known that they are all rationally equivalent. This can be seen in two ways. First, any point of $\overline{\mathcal{M}}_{g,n}$ corresponding to a curve with only rational components is in the image of the standard gluing map from $\overline{\mathcal{M}}_{0,2g+n}$ which identifies the $2g$ points in pairs. As $\overline{\mathcal{M}}_{0,2g+n}$ is rational, any two points on it are rationally equivalent. Another, purely combinatorial, method is to apply the standard linear equivalence on $\overline{\mathcal{M}}_{0,4}$ to give explicit rational equivalences between graphs related by the equivalence depicted in Figure 1. A simple combinatorial exercise shows that you can get between any two trivalent graphs using these moves. Thus it suffices to show that all 0-dimensional classes in the tautological ring are equivalent to sums of these.

3.2.

Top strata push forward to top strata under the natural morphisms (forgetting markings, and gluing maps, see (ii) and (iii) in Section 1.1) between $\overline{\mathcal{M}}_{g,n}$'s. The tautological ring is generated by intersection of ψ-classes, and pushforwards under these natural morphisms, so it suffices to show that all top intersections of ψ-classes are linear combinations of top strata.

3.3.

By Section 2.1, as all top intersection of ψ-classes are linear combinations of Hurwitz classes, it suffices to show that all Hurwitz classes are linear combinations of top strata.

3.4.

Consider a Hurwitz cycle $\mathbb{H}^g_\alpha$, corresponding to H^g_α covers of $\mathbb{P}^1$, with branching at $r+1$ given points of $\mathbb{P}^1$ (considered as an element of $\mathcal{M}_{0,r+1}$). Degenerate this element to the boundary consisting of a chain of $\mathbb{P}^1$'s, with 3 special points on each component (here we use the fact that all points of $\overline{\mathcal{M}}_{0,r+1}$ are rationally equivalent). Any cover of this configuration can have only rational components. This is because such a component maps to a $\mathbb{P}^1$ and is branched at at most three points, and one of those branch points is a simple branch point. It is an easy consequence of the Riemann-Hurwitz formula that such a cover is rational. Thus each component of the cover of the chain is rational, with two or three special points. The stabilization of such a curve has only rational components with three special points, and hence corresponds to a top stratum. $\square$

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