A TOOL FOR STABLE REDUCTION OF CURVES ON SURFACES

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ABSTRACT. In the study of the geometry of curves on surfaces, the following question often arises: given a one-parameter family of divisors over a pointed curve, what does the central fiber look like after stable or nodal reduction? We present a lemma describing the dual graph of the limit.

CONTENTS

1. INTRODUCTION

In the past decade, tremendous progress has been made in the enumerative geometry of nodal curves on surfaces. (The references include some of the most striking results.) Many of the results have been at least indirectly motivated by Kontsevich's introduction of stable maps, which was in turn motivated by mirror symmetry predictions from physics. These methods have been particular successful on surfaces, especially Fano and K3 surfaces, for many reasons.

The purpose of this expository note is to describe a result (Lemma 2.4.1) useful in the study of curves on a surface in a variety of contexts, especially in enumerative

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geometry. Although it does not appear to be in the literature, neither the statement nor its proof will surprise the experts. Nonetheless, the author has found it a useful tool to have ready at hand.

Suppose one wishes to study a family of curves on a surface S, such as plane curves with specified singularities, possibly satisfying certain incidence and tangency conditions. They can be studied from two points of view, (i) by considering the curves as divisors, i.e. as points in the Hilbert scheme, or (ii) by considering them as maps of nonsingular curves. Both have advantages; dimension counts and deformation theory have different flavors in the two cases, for example.

Certain questions naturally arise, such as the following.

1. Suppose one is given a morphism of a nodal curve to S. Can it be smoothed? In other words, can it be deformed to a map from a nonsingular curve, birational onto its image?
2. In the moduli space of stable maps, explicitly describe the boundary components of the component corresponding to immersions.
3. More generally, given a family of maps of nodal curves, what are the "codimension 1 degenerations"? What are the limit maps?
4. Suppose one is given an explicit family of divisors on S, parametrized by a pointed curve, whose general fiber is reduced. What does the central fiber look like after stable reduction?

The method of addressing all of these questions is the same: given a (possibly hypothetical) family of divisors on S, perform stable reduction. This is an explicit if tedious process (see [HM] Section 3E or [Ba] Section 1), but it is much more convenient to extract combinatorial information about the stable reduction by looking at the δ-invariants of the general fiber and central fiber.

2. Definition and result

In order to state the lemma in reasonable generality, we must generalize the classical notions of δ-invariants and dual graphs slightly.

2.1. **Loci of a divisor on a surface.**

Suppose D is a divisor on a nonsingular surface S. For the purpose of this note, a *locus* is defined to be the formal neighborhood of a closed subset L of S, including the data of the restriction of D, such that $\overline{D \setminus L}$ is reduced at $\overline{D \setminus L} \cap L$. A *branch* meeting a locus is a branch of $\overline{D \setminus L}$ meeting L. We will (sloppily) denote the locus by L also.

An example of a locus is a (formal neighborhood) of a connected component of $\operatorname{Sing} D$ (in S), which we will call a *singular locus.*

An example of a singular locus with two branches is given in Figure 1; in the figure, the "triple line" indicates that the component has multiplicity 3.

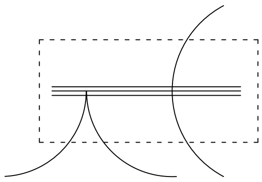

FIGURE 1. Example of a singular locus of a divisor on a surface

2.2. **δ-invariant of a locus.** To each locus L, we can associate a δ-*invariant*, specializing to the traditional definition when $\dim L = 0$. Specifically, define the curve L^c by normalizing the points of

$$\overline{D \setminus L} \cap L$$

on $\overline{D \setminus L}$ (i.e. the branches).

Then define

$$\delta_D(L) := p_a(D) - p_a(L^c).$$

The subscript D will be dropped when it is clear from the context.

2.2.1. *Remark.* Suppose $p \in L$, $\tilde{S} = \mathrm{Bl}_p S$, and $\tilde{D}$ (resp. $\tilde{L}$) is the total transform of D (resp. the preimage of the locus L). Then clearly $\delta_{\tilde{D}}(\tilde{L}) = \delta_D(L)$, as $p_a(D) = p_a(\tilde{D})$ and $p_a(L^c) = p_a(\tilde{L}^c)$.

2.2.2. *Computing* $\delta(L)$. One can compute $\delta(L)$ in general as follows. Blow up points of S repeatedly along points of L so that the total transform $\tilde{D}$ of D is a simple normal crossings divisor (not necessarily reduced!) along the preimage of L. By the preceding remark, this will not change the δ-invariant. Then it is easy to check that

$$\delta(L) = \sum_{n \text{ a node of } \tilde{L}} m_1 \cdot m_2 + \sum_{\substack{E \text{ irreducible} \\ \text{component of } \tilde{L}}} \left(\binom{m}{2} E^2 - m(1 - g(E)) \right)$$

where m_1 and m_2 are the multiplicities of $\tilde{D}$ along the branches of n, and m is the multiplicity of $\tilde{D}$ along E.

Example. By applying this procedure to the cusp, we obtain Figure 2, in which the special locus is indicated by the dashed box, components of the special locus are thick lines, the branch is a thin line, and multiplicities and self-intersections are indicated. (Each component has genus 0.) From this, we recover $\delta = 1$:

$$18 + 12 + 6 = 36$$

from the nodes, and

$$\left(\binom{6}{2}(-1) - 6 \right) + \left(\binom{3}{2}(-2) - 3 \right) + \left(\binom{2}{2}(-3) - 2 \right) = -35$$

from the irreducible components.

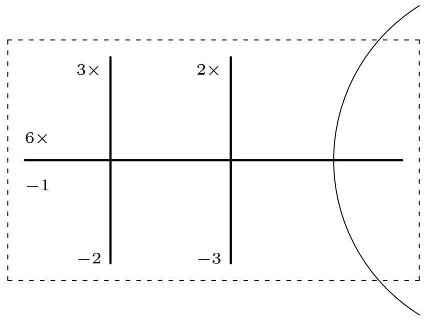

FIGURE 2. Computing the δ-invariant of a cusp

2.2.3. *Remark.* Suppose D is a divisor that is simple normal crossings on a locus L (but not necessarily reduced), and contains a one-dimensional component E. Then it is not hard to check that

$$\delta_D(L) - \delta_{D-E}(L) = E \cdot D - E^2 + g(E) - 1.$$

In particular, this difference is a numerical invariant (of the data (S, D, E, L)); this fact will be used in the third reduction step of the proof of Lemma 2.4.1.

2.3. **Locus of a map, and its dual graph.** Suppose $\rho : C \to S$ is a morphism from a nodal curve to a nonsingular surface. Let $[C]$ be the fundamental cycle of C, and let $\rho_*[C]$ be the image cycle in S. We define the *image divisor* of the map as the unique divisor in S whose associated cycle is $\rho_*[C]$. (*Warning:* this is not necessarily the scheme-theoretic image!) More generally, a family of maps of nodal curves to S over a base induces a morphism from the base to the Hilbert scheme.

To each locus L of D, associate the locus $\rho^{-1}(L)$ of C; call this a *locus* of the map ρ. It is a formal neighborhood of some union (not necessarily connected) of points and irreducible components. Define *branches* of this locus as before.

To each locus L of ρ, associate a *dual graph* Γ_L as follows. There is one vertex for each irreducible dimension one component of L, labelled with the arithmetic genus of that component, and a *single* vertex with no label representing *all* branches (even if there are *no* branches). Edges of the dual graph correspond to the nodes of the locus in the usual way.

An example of a locus of a map, and its corresponding dual graph, is given in Figure 3.

In analogy with the genus of the dual graph of a nodal curve, define the genus of dual graph to be

$$g(\Gamma_L) := 1 - \chi(\Gamma_L) + \sum \text{labels}$$

where $\chi(\Gamma_L)$ is the Euler characteristic, i.e. $h^0 - h^1$, or the number of edges less the number of vertices.

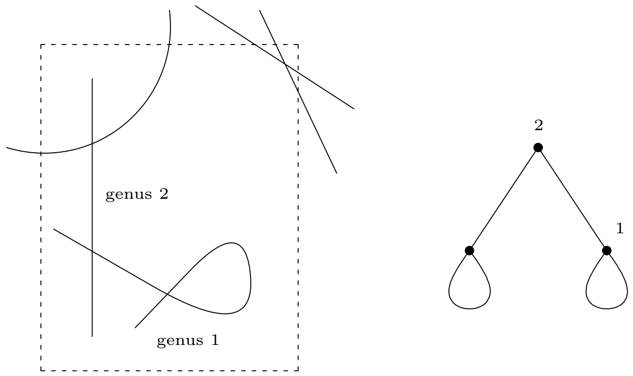

FIGURE 3. Example of a locus of a map, and its corresponding dual graph

2.4. **Statement of result.** Suppose

$$\begin{tikzcd} C \arrow[rr, "\rho"] \arrow[dr, "\pi"'] & & \mathcal{S} \arrow[dl] \\ & B & \end{tikzcd}$$

is a family of morphisms of maps of nodal curves to surfaces, where

- π is a (proper, flat) family of nodal curves whose general member is singular, and
- $\mathcal{S} \to B$ is a (proper, smooth) family of surfaces (which in applications will usually be taken to be the projection $S \times B \to B$).

Let 0 be a closed point of B, and η the generic point. For $b \in B$, we denote the fiber by $\rho_b : C_b \to \mathcal{S}_b$. Assume ρ_η maps C_η birationally onto its image. (This hypothesis can be removed, but it makes the proof simpler.)

Suppose L is a locus of the image divisor of C_0, with δ-invariant δ_0. (In practice, one often takes L to be a singular locus.) Let δ be the sum of the δ-invariants of the singular loci of the general fiber degenerating to L. Let g_0 be the genus of the dual graph of the corresponding locus on C_0.

Lemma 2.4.1. $\delta_0 - \delta = g_0$.

This makes precise the intuitive trade-off between "genus visible on the level of the Hilbert scheme" (i.e. δ-invariants) and "genus invisible on the level of the Hilbert scheme" (i.e. nodes and collapsed components).

3. MOTIVATION AND EXAMPLES

3.1. **Motivation.** The following are examples of results in the literature using stable or nodal reduction.

1. [H1] Proposition 3.1, and the similar argument in [H2] Lecture 3, Section 3.
2. [CH1] Proposition 2.5.
3. [CH3] Theorem 1.2.
4. [V2] Theorem 3.1.
5. [V3], the boundary divisor list of Section 4.2 (using Lemma 3.14 of [V1]).
6. [V4], the boundary divisor list of Theorem 4.2.

Part of the motivation was to give a uniform treatment of these arguments, and indeed they can all be replaced by easy applications of Lemma 2.4.1. Examples 1, 5, and 6 use the case where the locus in the central fiber is a point, while examples 2, 3, 4, and 6 use the case where the central locus is a curve.

In each case, the question is roughly one of describing the boundary components of a certain space of maps. Lemma 2.4.1 gives a geometric restriction on the maps that can appear, and a dimension bound ensures that the subvarieties satisfying the geometric restriction have codimension at least 1, with equality holding only for the desired boundary components.

A second motivation comes from a programme to prove Göttsche's conjecture, [G], following the approach of [Va] and [KP], except using stable maps in the place of the Hilbert scheme. A key step involves verifying that certain boundary divisors on a moduli space of stable maps are not "numerically relevant", which in turns requires verifying that the dimension of the space of various sorts of smoothable maps can be explicitly bounded. (*Note:* A.-K. Liu has announced a proof of Göttsche's conjecture.)

The third motivation was to have a convenient computational tool, see the examples below.

3.2. **Examples.**

3.2.1. *Nonsingular curves degenerating to a cusp.* If a family of nonsingular divisors on a surface degenerates to a curve with a cusp, it is well-known that the semistable limit has an elliptic tail, which is also immediate from Lemma 2.4.1. (However, explicit stable reduction shows that the elliptic tail must have j-invariant 0, which does *not* follow from Lemma 2.4.1.)

3.2.2. *Nonsingular curves degenerating to an arbitrary singularity.* More generally, if a family of nonsingular divisors degenerates to a curve with a singularity with δ-invariant δ_0 and b branches, the stable limit will have a contracted genus $\delta - b + 1$ component meeting all b branches. (This is not hard to show directly; it uses only the final step of the proof.) For the connectedness statement, an argument similar to that of the worked example (Section 4.1) can be used.

3.2.3. *Smoothability criterion.* The theorem also gives a criterion for smoothability. For example, a contracted curve of genus g attached to a branch of a curve mapping to a singularity with $\delta < g$ cannot be smoothed.

3.2.4. *Smoothing a contracted elliptic curve.* If a map from a contracted elliptic curve meeting two branches, whose images are tangent (i.e. the image has a tacnode), is smoothed (as a map), then nearby images will have a node. Thus the example of Section 4.1 is typical.

4. Proof of Lemma 2.4.1

The family induces a map from B to the Hilbert scheme; let

$$\begin{array}{cccc} \rho: & D & \hookrightarrow & \mathcal{S} \\ & \downarrow & & \\ & B & & \end{array}$$

be the corresponding universal closed immersion. Let L_0 be the locus of interest in the central fiber, and let $\mathcal{L}$ be the set of singular loci of the general fiber meeting (i.e. tending to) L_0.

The proof is by a series of reductions. For the convenience of the reader, a worked example follows the proof (Section 4.1). Throughout, we will pass to smaller neighborhoods of 0 in B, usually without comment. For example, we immediately discard the points of B (other than 0) whose fibers exhibit "nongeneric" singularities (i.e. that have combinatorially worse singularities than the general fiber). After our reductions, we will recover the family C over $B \setminus 0$, and the result will follow from examination of the central fiber.

First note that we can blow up sections (taking total transforms of the divisor D and the loci in question) without changing δ-invariants, by Remark 2.2.1.

Reduction 1. By blowing up sections that miss D away from the central fiber (shrinking B if necessary), and that meet D_0 at singularities contained in L, we can assume that D_0 is simple normal crossings along L_0, and that no two branches meet (i.e. all branches meet one-dimensional components of D_0).

Reduction 2. We next blow up sections contained in D so that the general fiber is simple normal crossings along the loci in $\mathcal{L}$, and so that no two branches meet in any locus in $\mathcal{L}$. To do this, note that we can do this on the generic fiber D_η by a series of blow-ups. By taking the closure in D, this series can be interpreted as a series of blow-ups along multisections. After base-change, the multisections become sections (for example, after base change the components and zero-dimensional singularities of the loci in $\mathcal{L}$ in the general fiber are distinguished, i.e. not exchanged by monodromy), and then the series is now a sequence of blow-ups of sections of D. Hence we can assume that all of the fibers of D are simple normal crossings along the loci in $\mathcal{L}$. Shrink B further by discarding fibers (other than 0) that are "combinatorially different" from the general fiber along the loci in $\mathcal{L}$.

Reduction 3. At this point, D_η may have multiple components (corresponding to exceptional divisors of the blowups of the previous reduction). Discard these components (one at a time). At each stage, this will change δ and δ_0 by the same amount by Remark 2.2.3. Thus we may assume that the fibers away from 0 are reduced, and that $\mathcal{L}$ is empty.

Final step. Thus we have reduced to the case $\delta = 0$, so we need only check that $\delta_0 = g$. Perform the stable (or nodal) reduction recipe on the locus L_0; this will involve base changing, as well as blow-ups and normalizations along (the preimage of) L_0. Let $D_0^\flat$ be the resulting central fiber, and let $L_0^\flat$ be the preimage of L_0 (which consists of nodal curves meeting branches of $\overline{D_0^\flat \setminus L_0^\flat}$, with dual graph Γ_L). Then a calculation of the arithmetic genus of the central fiber gives

$$p_a\left(D_0^\flat\right) = p_a\left(\overline{D_0^\flat \setminus L_0^\flat}\right) + \Gamma(L),$$

from which

$$\Gamma(L) = p_a\left(D_0^\flat\right) - p_a\left(\overline{D_0^\flat \setminus L_0^\flat}\right) = \delta_{D_0^\flat}(L_0^\flat)$$

as desired. □

4.1. **Worked example.** The following example may make the proof clearer. Consider the family

$$y^2 = t(x^2 + x^3) + x^6$$

parametrized by t, with special locus $(0,0)$ above $t = 0$. (Technically, we should take the closure in $\mathbb{P}^2$, but our constructions are all local.) Then the singularity in the central fiber is tacnode ($\delta_0 = 2$), and in the general fiber, there is one singularity (a node, $\delta = 1$) degenerating to the tacnode.

The results of the recipe given in the proof are shown in Figure 4. The location of the new exceptional divisor appearing after Reduction 2 depends on the choice of the section used in Reduction 1; for concreteness, we made a specific choice in the example.

Remark. Technically, we must also exclude the possibility that the dual graph is as given in Figure 5, i.e. that the image of the locus isn't connected, and a contracted genus 2 curve is attached to one of the branches, which is immersed. This can be shown in two ways. First, a genus 2 curve attached to an immersed branch can't be smoothed (see Section 3.2.3). Or second, if formally or étale-locally, there are two branches meeting with local intersection multiplicity two, nearby deformations will also have δ-invariant at least 2, contradicting $\delta = 1$.

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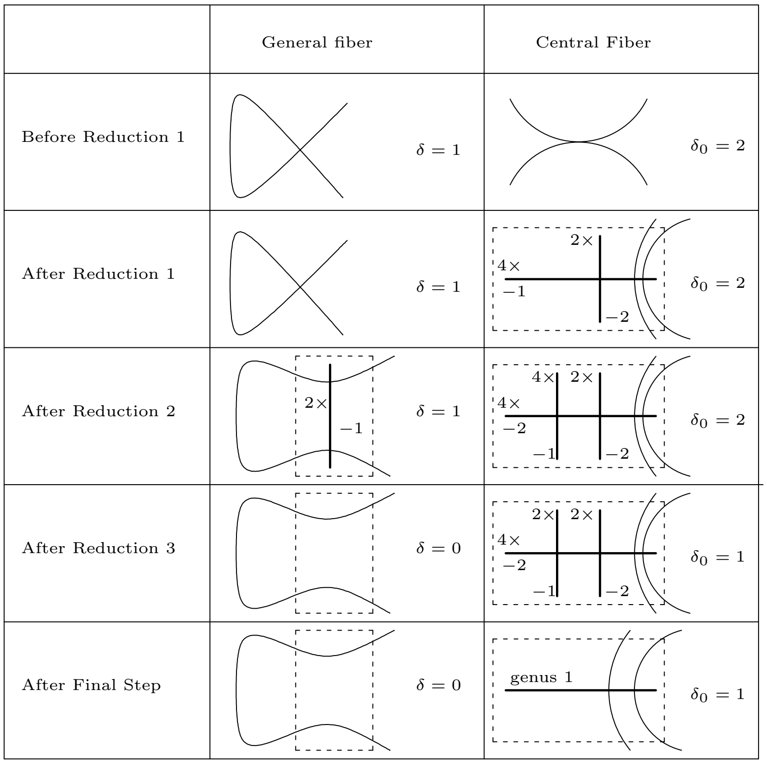

FIGURE 4. The worked example, after each step.

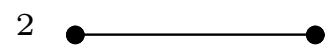

FIGURE 5. Not the dual graph of the locus on the central fiber in the worked example

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