GLUING SCHEMES AND A SCHEME WITHOUT CLOSED POINTS

KARL SCHWEDE

Abstract. We first construct and give basic properties of the fibered coproduct in the category of ringed spaces. We then look at some special cases where this actually gives a fibered coproduct in the category of schemes. Intuitively this is gluing a collection of schemes along some collection of subschemes. We then use this to construct a scheme without closed points.

1. Introduction

This paper is in essence a look at a naive attempt to glue schemes together. I define first a fibered coproduct in the category of ringed spaces. This particular coproduct is the natural generalization of gluing schemes along open subschemes. The coproduct is simply the pushout of the topological spaces combined with the appropriate pullback of the rings. This method, as we will see, does not always produce a scheme even when the ringed spaces involved are schemes and the morphisms between them are morphisms of schemes. However we will examine several cases where the coproduct is a scheme.

If we glue two affine schemes together using this method and the object along which we are gluing is a closed subscheme of one of the two schemes, the resulting coproduct is in fact an affine scheme (Theorem 3.3). Furthermore since every scheme is a ringed space we see that a fibered coproduct exists in the category of schemes at least in this case. This theorem has many immediate and perhaps unexpected consequences. First it allows us to glue two abstract schemes together along a common closed subscheme without first embedding the schemes in any ambient space. It allows us to contract any closed subscheme of an affine scheme over a field k, to a point. We do this by gluing the closed subscheme we wish to contract to a k-point. In particular, although it is well known that a line on $\mathbb{P}^2_k$ cannot be contracted, we will see explicitly that a line on $\mathbb{A}^2$ can be (even though the resulting scheme will not be noetherian).

A more specialized application of this method is gluing together a finite collection of closed (but not necessarily reduced) points of an affine scheme over a field k. This can be generalized to gluing together a finite collection of closed points in a quasi-projective variety because in that case every finite collection of points is contained in a single affine open subset. Specifically, if we glue two distinct single points of $\mathbb{A}^1_k$ to a single k-point using this method we get a nodal cubic. If we glue a double point $k[x]/(x^2)$ to a single k-point, we get a cuspidal cubic. This method gives us control over how the points are identified and lets us glue abstractly without first mapping the scheme into some projective space.

Another case where 3.3 can be directly applied is if we remove a part of an affine scheme via some localization (inverting certain elements). Then we can glue back a closed subscheme of the original affine scheme to recover some of the points that were removed (those that sat on the closed subscheme we glued back on). These points however have some topological oddities associated with them. In the resulting scheme we can only get to those recovered points along the scheme we glued back. See Proposition 2.6, Corollary 3.11, and Example 3.12.

Finally as a corollary of 3.3, we give an example of a scheme without closed points. We then look at an alternate construction of the same scheme using valuation rings. This second construction was also independently suggested to Arthur Ogus by Offer Gabber and the details were verified by Bjorn Poonen [2], but to my knowledge it has not been published.

This paper is self contained. For basic properties of schemes see [1]. Basic categorical definitions can be found in [3].

2. The coproduct in the category of ringed spaces and basic properties

We assume that all rings are commutative with unity and maps between rings send 1 to 1.

Suppose $\{X_i\}_{i\in I}$ is a collection of ringed spaces and for each (unordered) pair $i, j \in I$ there exists a ringed space $Z_{i,j}$ and morphisms of ringed spaces $\phi_{(i,j),i} : Z_{i,j} \to X_i$ and $\phi_{(i,j),j} : Z_{i,j} \to X_j$.

Definition We define the union of the X_i's along the $Z_{i,j}$'s (which we from now on will denote as $\cup_{Z_{s,t}} X_i$ or when there are only two sets to glue, $X \cup_Z Y$) as the set $\coprod X_i/\sim$ where the relation is generated by relations of the form $x_i \sim x_j$ ($x_i \in X_i$, $x_j \in X_j$) if there exists $z \in Z_{i,j}$ such that $\phi_{(i,j),i}(z) = x_i$ and $\phi_{(i,j),j}(z) = x_j$. Thus two points $x_i \in X_i$ and $x_j \in X_j$ are identified if and only if there exists a finite chain $x_{n_t} \in X_{n_t}$ where $t = 1 \dots m$ with $x_i = x_{n_1}$ and $x_j = x_{n_m}$ and for each pair (n_t, n_{t+1}) we have a $z_{n_t,n_{t+1}} \in Z$ such that $\phi_{(n_t,n_{t+1}),n_t}(z_{n_t,n_{t+1}}) = x_{n_t}$ and $\phi_{(n_t,n_{t+1}),n_{t+1}}(z_{n_t,n_{t+1}}) = x_{n_{t+1}}$ as above. Note that it is possible that $i = j$. We give it the strongest topology such that the natural maps α_s from X_s to $\cup_{Z_{i,j}} X_i$ are all continuous. We will now put a sheaf structure on the union. On each open $U \subset \cup_{Z_{i,j}} X_i$ note that $\alpha_i^{-1}(U)$ is an open subset of X_i and that $\phi^{-1}_{(i,j),i}(\alpha_i^{-1}(U)) = \phi^{-1}_{(i,j),j}(\alpha_j^{-1}(U))$. Define $\Gamma(U, \mathcal{O}_{\cup_{Z_{s,t}} X_i})$ to be the subring of the direct product $\prod_{i\in I} \alpha_i^{-1}(U)$ consisting of all tuples $(a_i)_{i\in I}$ such that $\phi^\sharp_{(i,j),i}(a_i) = \phi^\sharp_{(i,j),j}(a_j)$ (where in this case the $\sharp$ notation refers to the ring map portion of ϕ) for all pairs $i, j \in I$; in other words, the set of all sections of the X_i that agree on the Z_i. If I is empty we define $\cup_{Z_{s,t}} X_i$ as the empty scheme. ■

All we have done here is pushout the topological spaces and pullback the sheaf structure. We will now show that this is in fact a sheaf.

Proposition 2.1. *$\cup_{Z_{i,j}} X_i$ is a ringed space.*

Proof: Given the open sets $V \subset U \subset \cup_{Z_{i,j}} X_i$, the restriction map from U to V (on a tuple (a_i)) is just the restriction on each entry separately, that is $(a_i)_{i\in I}|_V = (a_i|_{\alpha_i^{-1}(V)})_{i\in I}$. It is clear that this satisfies the conditions of a presheaf. If U is an open subset of $\cup_{Z_{i,j}} X_i$, V_t is a covering of U and $s \in \mathcal{O}_{\cup_{Z_{i,j}} X_i}(U)$ is such that $s|_{V_t} = 0$ for all t it is clear the $s = 0$ since the X_i are sheaves. Likewise if U is again an open subset of $\cup_{Z_{i,j}} X_i$, $\{V_t\}$ is a covering of U and for each t there exists $s_t \in \mathcal{O}_{\cup_{Z_{i,j}} X_i}(V_t)$ such that $s_t|_{V_t\cap V_{t'}} = s_{t'}|_{V_t\cap V_{t'}}$ we need to show there exists $s \in \mathcal{O}_{\cup_{Z_{i,j}} X_i}(U)$ such that $s|_{V_t} = s_t$ for each t. But $s_t|_{V_t\cap V_{t'}} = ((s_t)_i|_{\alpha_i^{-1}(V_t\cap V_{t'})})_{i\in I} = ((s_t)_i|_{\alpha_i^{-1}(V_t)\cap\alpha_i^{-1}(V_{t'})})_{i\in I}$. So there exist $s_i \in \alpha_i^{-1}(U)$ such that $(s_i)|_{\alpha^{-1}(V_t)} = (s_t)_i$ since the X_i are sheaves. Therefore all we need to show is that $\phi^\sharp_{(i,j),i}(s_i) = \phi^\sharp_{(i,j),j}(s_j)$. However, that is easy because $(\phi^\sharp(s_i) - \psi^\sharp(s_j))|_{\phi^{-1}_{(i,j),i}(\alpha_i^{-1}(V_t))} = \phi^\sharp_{(i,j),i}((s_t)_i) - \phi^\sharp_{(i,j),j}((s_t)_j) = 0$ since s_t was defined that way, the $Z_{i,j}$ are sheaves, and the inverse images of the V_t's cover the inverse image of U. ■

This gives us morphisms of ringed spaces from the X_i to $\cup_{Z_{i,j}} X_i$ (via the α_i) where (on the rings) we simply project to each coordinate. In the same way we get morphisms $\gamma_{i,j}$ from $Z_{i,j}$ to $\cup_{Z_{i,j}} X_i$, those being the composition maps $\alpha_i \circ \phi_{(i,j),i} = \alpha_j \circ \phi_{(i,j),j}$.

This procedure does not in general produce a scheme even if the X_i and $Z_{i,j}$ are schemes. Furthermore, even when it does it may not produce a noetherian scheme even if the all the schemes being glued together are noetherian.

The advantage of this definition is that it immediately gives us the following universal property.

Theorem 2.2. *Suppose that W is a ringed spaces and there exists maps $\psi_i : X_i \to W$ for each $i \in I$ such that each square of the form*

$$\begin{tikzcd}
 & Z_{i,j} \arrow[dl, "\phi_{(i,j),i}"'] \arrow[dr, "\phi_{(i,j),j}"] & \\
X_i \arrow[dr, "\psi_i"'] & & X_j \arrow[dl, "\psi_j"] \\
 & W &
\end{tikzcd}$$

commutes. Then the maps ψ_i factor through the natural maps α_i to $Y = \cup_{Z_{i,j}} X_i$, so that the following diagrams commute for each $i, j \in I$.

$$\begin{tikzcd}
 & Z_{i,j} \arrow[dl, "\phi_{(i,j),i}"'] \arrow[dr, "\phi_{(i,j),j}"] & \\
X_i \arrow[dr, "\alpha_i"] \arrow[ddr, bend right, "\psi_i"'] & & X_j \arrow[dl, "\alpha_j"'] \arrow[ddl, bend left, "\psi_j"] \\
 & X_i \cup_{Z_{i,j}} X_j \arrow[d] & \\
 & W &
\end{tikzcd}$$

Proof: The proof of this fact is an easy consequence of the definition. ■

Let us first record some basic properties of this construction. First we will show that gluing X to Y along X simply gives us Y again.

Proposition 2.3. *Suppose that X and Y are ringed spaces and $\psi : X \to Y$ is a morphism. Then $X \cup_X Y \cong Y$ where the map from X to X is the identity.*

Proof: As sets Y and $X \cup_X Y$ are naturally identified by the map β. They are homeomorphic since the topology on $X \cup_X Y$ is the strongest that make α and β continuous. On an open subset U of $X \cup_X Y = Y$ the sections are the pairs (f, g) where $f \in \mathcal{O}_X(\psi^{-1}(U))$ and $g \in \mathcal{O}_Y(U)$ such that $f = \psi^\sharp(g)$. Thus the sections are the pairs $(\psi^\sharp(g), g)$ which is isomorphic to $\mathcal{O}_Y(U)$. This completes the proof. ■

Now we will prove some results which will shed some light on the topological structure of this object.

Lemma 2.4. *The open (closed) subsets $\cup_{Z_{i,j}} X_i$, $i \in I$ correspond bijectively to tuples $(U_i)_{i\in I}$ of open (closed) subsets of the X_i such that $\phi^{-1}_{(i,j),i}(U_i) = \phi^{-1}_{(i,j),j}(U_j)$*

Proof: Clearly an open subset of $\cup_{Z_{i,j}} X_i$ gives a natural tuple of open subsets of the X_i that agree on the $Z_{i,j}$ (their preimages). On the other hand, if (U_i) is such a tuple we need to show that $W = \cup \alpha_i(U_i)$ is an open subset of $\cup_{Z_{i,j}} X_i$. Now clearly $U_i \subset \alpha_i^{-1}(W)$ so we need to prove equality. The following is essentially a small lemma. Suppose $x \in X_i$ and there exists $z \in Z_{i,j}$ such that $\phi_{(i,j),i}(z) = x$. Let us also denote $\phi_{(i,j),j}(z)$ by x_j and finally suppose that $x_j \in U_j$. I will show that $x \in U_i$. But $z \in \phi^{-1}_{(i,j),j}(U_j) = \phi^{-1}_{(i,j),i}(U_i)$ so $x = \phi_{(i,j),i}(z) \in \phi_{(i,j),i}(\phi^{-1}_{(i,j),i}(U_i))$ which is contained in U_i. Now for the more general case. Suppose $x \in X_i$ and $\alpha_i(x) \in W$. Then there exists $j_1, \ldots, j_n$ with $j_1 = i$ and $z_{j_m,j_{m+1}} \in Z_{j_m,j_{m+1}}$ where $\phi_{(j_m,j_{m+1}),j_{m+1}}(z_{j_m,j_{m+1}}) = \phi_{(j_{m+1},j_{m+2}),j_{m+1}}(z_{j_{m+1},j_{m+2}})$, $\phi_{(j_1,j_2),j_1}(z_{j_1,j_2}) = x$ and $\phi_{(j_{n-1},j_n),j_n}(z_{j_{n-1},j_n}) \in U_{j_n}$. Finally let us denote $\phi_{(j_{n-1},j_n),j_n}(z_{j_{n-1},j_n})$ as x_n so we have by defintion $\alpha_{j_n}(x_n) = \alpha_{j_1}(x) \in W$. Using induction and the previous statement we conclude that $x \in U_i = U_{j_1}$. ∎

Lemma 2.5. *Let $\phi : Z \to X$ and $\psi : Z \to Y$ be morphisms of ringed spaces and let $\alpha : X \to X \cup_Z Y$ and $\beta : Y \to X \cup_Z Y$ be the induced maps. Suppose that ϕ (or ψ) is a homeomorphism onto its image. Then so is β (or α).*

Proof: First note that when ϕ is injective (as a map of topological spaces) there is no collapsing of points of Y (which can happen in general) so at least β is an injective map. The map β is continuous by definition so all we need to show is that if $U \subset Y$ is open there exists an open $V \subset X \cup_Z Y$ such that $\beta^{-1}(V) = U$. So given U as above, since ϕ is a homeomorphism onto its image there exists $U' \subset X$ such that $\psi^{-1}(U) = \phi^{-1}(U')$. Now consider $\alpha(U) \cup \beta(U')$. This is an open subset of $X \cup_Z Y$ (by lemma 2.4) so it satisfies the desired property. ∎

Proposition 2.6. *Let X, Y and Z be as in the previous lemma with the maps between them labelled in the same way. Now suppose that ψ is a homeomorphism onto its image and that Z is a Zariski space (noetherian and every irreducible closed set has a unique generic point). Then if $x \in X$ and $y \in Y$ and if $\beta(y) \in \{\alpha(x)\}^-$ then there exists $z \in Z$ such that $y \in \{\psi(z)\}^-$ and $\phi(z) \in \{x\}^-$.*

Proof: First note that since ψ is a homeomorphism onto its image, α is also, which implies that $\{x\}^- = \alpha^{-1}(\{\alpha(x)\}^-)$. Now look at $\phi^{-1}(\{x\}^-)$. We can assume this is nonempty because if it were empty then $\alpha(\{x\}^-) \cup \beta(\emptyset)$ would be closed, in which case $\beta(y) \in \alpha(\{x\}^-)$. But the points of $\{x\}^- \in X$ cannot be identified with any points of Y since $\phi^{-1}(\{x\}^-) = \emptyset$ which is a contradiction. Since ψ is a homeomorphism onto its image, every closed subset of Z arises as the inverse image of a closed subset of Y, (including the inverse images of closed subsets of X). Then because $\beta(y)$ is in the closure of $\alpha(x)$, by 2.4, for every closed subset of $V \subset X$ containing x, each closed subset (there exists at least one) W of Y such that $\psi^{-1}(W) = \psi^{-1}(V)$ contains y. In particular $\psi(\phi^{-1}(\{x\}^-))^-$ contains y. Since Z is Zariski and $\phi^{-1}(\{x\}^-)$ is closed we can write $\phi^{-1}(\{x\}^-) = Z_1 \cup \ldots \cup Z_n$ for irreducible closed sets Z_i with generic points $z_i \in Z_i$. Thus $y \in \psi(Z_1 \cup \ldots \cup Z_n)^- = \psi(Z_1)^- \cup \ldots \cup \psi(Z_n)^-$ which implies that $y \in \psi(Z_i)^-$ for some i. Since continuous maps preserve specialization (points being in the closure of other points) we conclude that $y \in \{\psi(z_i)\}^-$. On the other hand $\phi(z_i) \in \{x\}^-$, which completes the proof. ■

We will now prove a result saying we do not have to glue all at once.

Proposition 2.7. *Let X_i and $S = \{Z_{i,j}\}$ be as in definition at the start of the section. Suppose $I = I' \cup I''$ and $I' \cap I'' = \emptyset$. Let S' be the subset of S where both indices of $Z_{i,j}$ are in I' and let S'' be the subset of S where both indices of $Z_{i,j}$ are in I''. Let $S_0 = S - (S' \cup S'')$. Let $Z = \coprod_{S_0} Z_{i,j}$ (as a topological space this is just the disjoint union and on the rings we simply take direct products). Let $X' = \cup_{S'} X_i$ where in this union the X_i are indexed by I'. Likewise let $X'' = \cup_{S''} X_i$. Then there exist maps $\phi' : Z \to X'$ and $\phi'' : Z \to X''$ induced by the $Z_{i,j}$'s making up Z and $X' \cup_Z X'' \cong \cup_{Z_{i,j}} X_i$.*

Proof: This is a direct consequence of the definition. ■

Obviously we could take more general partitions as well. This proposition is the computational tool I will use to compute all examples where more than two schemes are glued to together.

3. An Application to Schemes

We will now use this construction to glue schemes together. Unless otherwise noted, for the rest of the paper X, Y, and Z will be schemes, ϕ will be a map from Z to X and ψ will be a map from Z to Y.

Before we actually do any computations we need to make one more observation. Suppose that we have X, Y and Z as above and there exist open $U \subset X$, $V \subset Y$ such that $W = \phi^{-1}(U) = \psi^{-1}(V)$. Then $U \cup_W V$ is isomorphic to $(X \cup_Z Y)|_{\alpha(U)\cup\beta(V)}$. Topologically this is clear (note what the open subsets of $U \cup_W V$ are). The sections are naturally identified as well.

Example 3.1. (Gluing along open sets) Suppose ϕ and ψ are open immersions. Then $X \cup_Z Y$ is easily seen to be the standard gluing of X and Y along Z and thus a scheme. ■

Example 3.2. (Gluing along an open/closed subscheme) Suppose $X = \mathbb{A}^2_k = \operatorname{Spec} k[x, y]$ and $Z = X - \{0\}$ where 0 is the origin. Also let $Y = \mathbb{A}^3_k - \{0\} = \operatorname{Spec} k[x, y, z] - \{0\}$. Let the maps ϕ and ψ be the natural ones, where ϕ is an open immersion and ψ is a closed immersion. We will show that $X \cup_Z Y$ is not a scheme. Let P be the origin of X (the point missed by ϕ). Each neighborhood of P in $X \cup_Z Y$ corresponds to a pair of open sets $U \subset X$ and $V \subset Y$ such that $\phi^{-1}(U) = \psi^{-1}(V)$. But for any such pair with $P \in U$, V cannot be affine and the prime spectrum of the sections of V will contain the missing origin point. Note that the sections of $U \subset X$ and $\phi^{-1}(U) \subset Z$ are isomorphic for each U so gluing to X neither adds nor removes any new sections besides those associated with Y, i.e. for each open set $U \cup_W V \subset X \cup_Z Y$, the sections $\mathcal{O}_{X\cup_Z Y}(U \cup_W V) = \mathcal{O}_Y(V)$. However, the topology on $X \cup_Z Y$ is too strong for this to be a scheme since not every line (which we think of going through the origin in $\mathbb{A}^3$) actually contains the origin of $X \cup_Z Y$ in its closure as per proposition 2.6. In particular the line Z corresponding to $z = 0$ in $Y \subset \mathbb{A}^3$ does not contain P in it's closure in $X \cup_Z Y$ since $\phi^{-1}(\emptyset) = \psi^{-1}(Z)$. So that the line minus the point is a closed set. ■

The next theorem is the main theorem of the section. It provides a condition for when we can glue schemes together and get a scheme. We will see that this works in many special cases as well. It also will be important for one construction of a scheme without closed points.

Theorem 3.3. *Suppose A and B are rings. Further suppose I is an ideal of A and there exists a map γ from B to A/I. We will denote the quotient map from A to A/I by π. Let $X = \operatorname{Spec} A$, $Y = \operatorname{Spec} B$ and $Z = \operatorname{Spec} A/I$, so that Z is a closed subscheme of X. Then $X \cup_Z Y$ is an affine scheme with Y a closed subscheme, $(X \cup_Z Y) - Y \cong X - Z$, and the maps $\alpha : X \to X \cup_Z Y$ and $\beta : Y \to X \cup_Z Y$ are morphisms of schemes.*

Informally we are going to glue $Y = \operatorname{Spec} B$ onto X along Z (the map from Z to Y is the one induced by γ). This will replace Z with Y (whatever that might mean) while keeping $X \cup_Z Y$ an affine scheme.

Proof: First let us look at what $X \cup_Z Y$ will look like as a set. Since $\phi : Z \to X$ is a homeomorphism onto its image $\beta : Y \to X \cup_Z Y$ is also. Since Z is a closed subset of X and we are gluing along Z, we see that X must remain the same outside of Z. Furthermore Z will be replaced by Y via the map $\psi : Z \to Y$ induced by γ.

Since I am claiming $X \cup_Z Y$ is affine let us look first at the global sections. The global sections are $C = \{(a, b) \mid a \in A, b \in B, a + I = \gamma(b)\}$. Let the maps $C \to A$, $C \to B$ induced by restricting to each coordinate be denoted by f and g respectively. Let $J = \{(a, 0) \in C\} = \ker g$ and let $J' = \{(0, b) \in C\} \ker f$. It is then easy to see that $C/J \cong B$ with this isomorphism being induced by g. We can view C/J' as a subring of A and in fact it can be described as $\pi^{-1}(\gamma(B))[I]$, the $\pi^{-1}(\gamma(B))$ subalgebra of A generated by I. In the future we will denote $\pi^{-1}(\gamma(B))$ as B' and $\operatorname{im}(C/J') \subset A$ as $B'[i]$.

Let P be a prime ideal of C. Since $JJ' = (0)$, P must contain either J or J'. If P contains J then P corresponds via g to a prime ideal in B. On the other hand if P does not contain J then it must contain J', so it corresponds to a prime ideal Q' of $B'[I]$. Since it didn't contain J it cannot contain I since $J = \{(I, 0)\}$. Thus there exists $a \in I$, $a \notin Q'$. Then Q' corresponds to a prime ideal of $B'[I][a^{-1}] = A$ since $a \in I$. So P corresponds to a prime ideal Q of A. Note that by the naturality of this chain we have $f^{-1}(Q) = P$ and the Q satisfying this property is unique. Thus at least as a set, $\operatorname{Spec} C$ corresponds to $(X - Z) \cup Y$ and the maps f and g induce the expected correspondences.

Now we need to show that this correspondence is in fact a homeomorphism of topological spaces. Let W be a closed subset of the topological space of $X \cup_Z Y$. Thus $\alpha^{-1}(W)$ and $\beta^{-1}(W)$ are closed subsets of X and Y respectively so they are cut out by ideals $K \subset A$ and $K' \subset B$. Let $L = f^{-1}(K) \cap g^{-1}(K') \subset C$. I will show that the points cut out by L are precisely the points of W. First suppose P is a prime ideal of C corresponding to a point of W. Then P comes from a point in X or from a point in Y so it is either a point of $\alpha^{-1}(W)$ or of $\beta^{-1}(W)$. Thus the appropriate corresponding prime of A or B contains either K or K' so it follows that P must contain L. On the other hand, suppose P is a prime ideal of C containing L. Then P must contain $f^{-1}(K)$ or $g^{-1}(K)$. If P contains $g^{-1}(K')$ then it must also contain $J = \ker g \subset g^{-1}(K')$ so that P corresponds to an element of $\beta^{-1}(W)$. If P contains $f^{-1}(K)$ and does not contain J then P corresponds to a prime of A containing K, that is an element of $\alpha^{-1}(W)$. The one case we must worry about is if P contains both $f^{-1}(K)$ and J. In this case I will show that P contains $g^{-1}(K')$ which puts P in the first case again. In particular it is enough to show that $K' \subset g(P)$ since g surjects and P contains $J = \ker g$. Note that $\sqrt{\gamma(K')A/I} = \sqrt{\pi(K)}$ since the inverse images of W in Z must be the same whether we take inverse images through X or Y. Now take $b \in K' \subset B$, then

$\gamma(b) \in \gamma(K') \subset \sqrt{\pi(K)}$ so $\gamma(b^n) \in \pi(K)$ for some n. Choose a representative $a + I = \gamma(b^n)$ so that $a \in K$. Then $(a, b^n) \in C$. Since $a \in K$ and $b^n \in K'$ we have $(a, b^n) \subset L$ which implies $(a, b^n) \in P$ so $b^n \in g(P)$ and since $g(P)$ is still prime we conclude $b \in g(P)$ as desired. Therefore, since the topology on $X \cup_Z Y$ was chosen to be the strongest possible, the correspondence we established between $\operatorname{Spec} C$ and $X \cup_Z Y$ is a homeomorphism.

Next we need to show that $X \cup_Z Y$ and $\operatorname{Spec} C$ are isomorphic as schemes. We need only work on the affine open sets $\operatorname{Spec} C_c$ since these sets form a basis. If $c = (s,t) \in C$ then we need to show that the natural map of C_c to the global sections of $(X - V(s)) \cup_{(Z - V(s+I))} (Y - V(t))$ is an isomorphism. This map is $(a,b)/c^n \mapsto (a/s^n, b/t^n)$. First let us show that it is injective. Suppose $(a/s^n, b/t^n) = 0 = (0,0)$, so there exists m_1, m_2 such that $s^{m_1} a = 0$ and $t^{m_2} b = 0$. But then we can choose m large enough so that $c^m(a,b) = 0$ so $(a,b)/c^n = 0$ as well. To show surjectivity take $(a/s^n, b/t^m)$ and note that either by scaling a or b by s or t respectively we can assume $n = m$. So we have $(a/s^n, b/t^n)$. But $\pi(a)/\pi(s)^n = \gamma(b)/\gamma(t)^n$ and since $\pi(s) = \gamma(t)$ we see that there exists l such that $\pi(s)^l(\pi(a) - \gamma(b)) = 0$ so $\pi(as^l) = \gamma(bt^l)$ for some l. Then $(as^l, bt^l)/c^{n+l}$ is mapped to $(a/s^n, b/t^n)$ as desired. Since these maps were chosen to be compatible with the restriction maps, we have $X \cup_Z Y \cong \operatorname{Spec} C$. Furthermore, $\operatorname{Spec} C$ has Y as a closed subscheme and is isomorphic to $X - Z$ outside of it.

Finally since the maps between the ringed spaces $\alpha : X \to X \cup_Z Y$ and $\beta : Y \to X \cup_Z Y$ are induced by maps of the global sections of these affine schemes we see that α and β are morphisms of schemes as desired. ■

Corollary 3.4. (Contracting a closed set to a point in an affine scheme) *Let A, I, and B be as above and further assume that A is a k algebra and that $B = k$. Then $X \cup_Z Y$ is isomorphic to* $\operatorname{Spec} A$ *outside of* $\operatorname{Spec} A/I$ *and the closed subscheme* $\operatorname{Spec} A/I$ *is contracted to a point.*

Proof: Direct from 3.3. ■

Example 3.5. (Contracting a line in $\mathbb{A}^2_k$). Let $A = k[x,y]$, $I = (x)$, $B = k$, then $X \cup_Z Y = \operatorname{Spec} k[x, xy, xy^2, xy^3, \ldots]$. This is $\mathbb{A}^2_k$ with the line $x = 0$ contracted to a point. Note that in this case the resulting scheme is not noetherian even though the originals were. ■

Example 3.6. (Gluing points on $\mathfrak{A}^1_k$) Let $A = k[x]$ and let $I = ((x-1)(x+1))$; then $X \cup_Z Y$ is easily seen to be isomorphic to the nodal cubic. If $I = (x^2)$ then $X \cup_Z Y$ is the cuspidal cubic. ■

We are going to use 3.3 to glue along closed subschemes of arbitrary schemes.

Corollary 3.7. (Gluing closed subschemes in general) *Suppose Z is a closed subscheme of both X and Y. Then $X \cup_Z Y$ is a scheme.*

Proof: Outside of Z we must have a scheme, so choose $x \in X$ in the image of Z with $z \in Z$ such that $\phi(z) = x$. Choose an affine open $U \subset X$ with $x \in U$. Then $\phi^{-1}(U)$ is an open affine subset of Z. Since $\psi : Z \to Y$ is a homeomorphism onto its image there exists an open $W \subset Y$ such that $\psi^{-1}(W) = \phi^{-1}(U)$. Choose an affine subset V of W such that $\psi(z) \in V$. Then $\psi^{-1}(V)$ is an affine subset of $\phi^{-1}(U)$. Then there exist further affine subsets (localizations in fact from U and V) $V' \subset V$, $U' \subset U$ such that $z \in \phi^{-1}(U'), \psi^{-1}(V')$ and $\phi^{-1}(U') = \psi^{-1}(V')$. Then $U' \cup_{\phi^{-1}(U')} V'$ is an open neighborhood of x in $X \cup_Z Y$. But by 3.3, that is affine. Thus $X \cup_Z Y$ is locally affine so it is a scheme. ■

In fact it is not hard to see that if X and Y are closed subschemes of an ambient scheme and if Z is the scheme corresponding to the intersection (whose ideal sheaf is the sum of X and Y's ideal sheaves) then $X \cup_Z Y$ in fact corresponds to the scheme cut out by the intersection of the ideal sheaf of X with the ideal sheaf of Y. At this point it would be natural to wonder whether we can glue more than two schemes together (perhaps along closed subsets) and still get a scheme. We shall next give an example when this does indeed happen, although not always in the way we expect. First we need a lemma that improves 3.3

Lemma 3.8. *Suppose $X = \operatorname{Spec} A$ and $Y = \operatorname{Spec} B$ are affine schemes and suppose the scheme $Z = \operatorname{Spec} C$ maps to them via the maps ϕ and ψ as before. Further suppose* $\operatorname{im} A \subset \operatorname{im} B \subset C$ *and that ψ is a closed map (of topological spaces). Then $X \cup_Z Y$ is an affine scheme with X as a closed subscheme.*

Proof: Let $W = \operatorname{Spec}(\operatorname{im} B)$. Immediately we notice that we have the following diagram.

$$\begin{tikzcd}
 & Z \arrow[d, "\delta"] \arrow[ddl, bend right, "\phi"'] \arrow[ddr, bend left, "\psi"] & \\
 & W \arrow[dl, "\phi'"] \arrow[dr, "\psi'"'] & \\
X \arrow[dr, "\alpha"'] & & Y \arrow[dl, "\beta"] \\
 & X \cup_Z Y &
\end{tikzcd}$$

where the map δ is induced by the inclusion of $\operatorname{im} B$ into C, thus W is just the closure of the image of Z in X, but since ψ is closed this is just the image of ψ, and so δ is surjective as well. First let us show that $X \cup_W Y$ and $X \cup_Z Y$ are the same as topological spaces. Since δ is surjective no additional relations are added and by the factorization all original relations are kept; thus $X \cup_W Y$ and $X \cup_Z Y$ are identified as sets. To see that they are identified topologically too, we recall 2.4 and note that if $U \subset X$ and $V \subset Y$ are open subsets such that $\phi^{-1}(U) = \psi^{-1}(V)$ then $\phi'^{-1}(U) = \delta(\phi^{-1}(U)) = \delta(\psi^{-1}(V)) = \psi'^{-1}(V)$ again since δ is surjective. Likewise if $\phi'^{-1}(U) = \psi'^{-1}(V)$ we have that $\phi^{-1}(U) = \delta^{-1}(\phi'^{-1}(U)) = \delta^{-1}(\psi'^{-1}(U)) = \psi^{-1}(U)$. Now we will show that they are isomorphic as sheaves. Choose an open subset of $X \cup_W Y$ corresponding to a pair $U \subset X$, $V \subset Y$. Then the sections of this are the sections that agree in W. But since $\delta^\sharp$ is an injective map of sheaves, these are just the sections that agree in Z as desired. However now we are in the case of 3.3 so $X \cup_Z Y = X \cup_W Y$ is an affine scheme. ∎

We will now see that if we have a collection of closed subsets of a scheme we can glue them together along their intersections to get a scheme. However, as we will see, this scheme is not always a closed subscheme of the ambient space, although it does always map there (via the universal property).

Theorem 3.9. *Suppose Y is a scheme and $X_1, \ldots, X_n$ are closed subschemes. Let us denote by $Z_{i,j}$ the closed subscheme corresponding to the intersection of X_i and X_j. Then $\cup_{Z_{i,j}} X_i$ is a scheme and the X_i are closed subschemes.*

Proof: Without loss of generality we may assume that Y is affine (say equal to $\operatorname{Spec} A$) since we can always restrict. Let us denote by I_i the ideal corresponding the X_i so that $Z_{i,j} = \operatorname{Spec}(A/(I_i+I_j))$. We will proceed by induction on n. The base case is clear so suppose we can glue up to n closed subschemes. Let X_{n+1} be another closed subscheme corresponding to an ideal I_{n+1}. Let us denote $X = (\cup_{Z_{i,j},1\leq,i,j\leq n} X_i) = \operatorname{Spec} B$ and $Z = \coprod_{i=1}^n Z_{i,n+1} = \operatorname{Spec} C = \operatorname{Spec} \oplus_{i=1}^n A/(I_i + I_n + 1)$. Then by proposition 2.7 we have $\cup_{Z_{i,j},1\leq,i,j\leq n+1} X_i = X_{n+1} \cup_Z X$. By the universal property 2.2 we see that there are maps $A \to B \to C$ however we also have $A \to A/(I_{n+1}) \to C$ with the first map surjective. Thus $\operatorname{im} A/(I_{n+1} \subset \operatorname{im} B \subset C$ so we can apply lemma 3.8 which shows $\cup_{Z_{i,j}} X_i$ is a scheme. The lemma also guarantees that each X_i is a closed subscheme since the choice of X_{n+1} was arbitrary. ∎

Example 3.10. As noted the scheme constructed this way is not always a subscheme of the ambient space (Y in the notation of 3.9). For example let $Y = \mathbb{A}^2 = \operatorname{Spec} k[x,y]$, $X_1 = \operatorname{Spec} k[x,y]/(x)$, $X_2 = \operatorname{Spec} k[x,y]/(y)$, and $X_3 = \operatorname{Spec} k[x,y]/(x-y)$. Now all of the $Z_{i,j}$'s are just $\operatorname{Spec} k = \operatorname{Spec} k[x,y]/(x,y)$. If we glue all three together simultaneously we get a scheme isomorphic to $\operatorname{Spec} k[x,y,z]/(x,y)\cap(x,z)\cap(y,z)$ which is not equal to $\operatorname{Spec} k[x,y]/(x)\cap(y)\cap(x-y)$ since the dimension of the tangent space at the intersection point of the first is three and the dimension at the intersection point of the second is two. However if we replace the X_i's by schemes with embedded points at the intersection points, that is $X_1 = \operatorname{Spec} k[x,y]/(x^2,xy)$, $X_2 = \operatorname{Spec} k[x,y]/(y^2,xy)$ and $X_3 = \operatorname{Spec} k[x,y]/((x-y)^2, x^2-y^2)$ the scheme obtained by gluing all three simultaneously is $k[x,y,z]/(x)\cap(y)\cap(x-y)$. ∎

We will now continue on with some other corollaries of 3.3

Corollary 3.11. (Gluing a closed subset of a $\operatorname{Spec} R$ onto $\operatorname{Spec} S^{-1}R$) *Let R be a ring, S a multiplicative subset of R, and J an ideal of R. Then we let $A = S^{-1}R$, $B = R/J$, and let the ideal I of A be $S^{-1}J$. Then $X \cup_Z Y$ can be identified (as sets) as the union of* $\operatorname{Spec} S^{-1}R$ *and R/J.*

Note that while we can make this identification as sets, the topology may be stronger than we might expect. We see this in the next example.

Example 3.12. As in the above corollary let $R = k[x,y]$, $S = \{1, y, y^2, \ldots\}$ and let $I = (x)$. Then $X \cup_Z Y = \operatorname{Spec} k[x, y, \frac{x}{y}, \frac{x}{y^2}, \ldots]$ and this scheme looks like $\mathbb{A}^2$ minus the line $y = 0$ but with the origin put back in (actually the whole line $x = 0$ was put in, but most of it was already there). However, there is a topological pathology created. Proposition 2.6 tells us that the only curves going through the origin must now contain $x = 0$ in their closure. For example the line corresponding to $x - y = 0$ now misses the origin. We can see this algebraically since the ideal $(x-y) = (y)(\frac{x}{y} - 1)$ and $(\frac{x}{y} - 1)$ which is the corresponding prime ideal clearly doesn't go through the origin $m = (x, y, \frac{x}{y}, \frac{x}{y^2}, \ldots) \lhd k[x, y, \frac{x}{y}, \frac{x}{y^2}, \ldots]$. If we took $I = (y)$ instead in an attempt to glue back the line we removed, we notice that Z becomes the empty scheme, so we have only the disjoint union of the plane minus a line and the line. ∎

Finally let us look at a related example where we do not get a scheme.

Example 3.13. Let $X = \operatorname{Spec} k[x,y]_y$, $Y = \operatorname{Spec} k[x,y]_{(x,y)}$, and $Z = \operatorname{Spec}(k[x,y]_{(x,y)})[y^{-1}]$. Thus X is the plane without a line, Y is the local ring of the origin, and Z is the local ring of the origin with the line $y = 0$ missing and thus of course the original (closed) point missing as well. Let the maps ϕ and ψ be those induced by inclusions. None of these maps are surjective so 3.3 does not apply. Now on any open neighborhood of the origin in $X \cup_Z Y$, the corresponding open subset of Y must be all of Y. So y must not be invertible in that open set. Therefore we pick up at the very least an open dense subset of the line $y = 0$ in the plane $\mathbb{A}^2$. None of these points (except the origin itself) are in $X \cup_Z Y$, so we do not have a scheme. Note in this case the global sections are just $k[x, y]$. ∎

4. A scheme without closed points

In this section we will give two constructions of a scheme without closed points. We will utilize the gluing methods from the previous section, the other will use valuation rings. The two schemes we get in this section are easily seen to be the same.

Proposition 4.1. *If X is a quasi-compact scheme then X has a closed point.*

Proof: Since X is quasi-compact there is a cover by affine $\{U_i\}$, $U_i = \operatorname{Spec} A_i$. Take a maximal ideal (closed point) P_1 of U_1. If P_1 is closed in X we are done. If not, take P_2 to be any point (besides P_1 itself) in its closure. Now P_2 is in some U_i (but not U_1) so without loss of generality say P_2 is in U_2. If P_2 is closed we are done. If not take a point P_3 in its closure. Again P_3 is in one of the U_i's but this time it cannot be in either U_1 or U_2 (since it is in the closure of both P_1 and P_2), so without loss of generality we say $P_3 \in U_3$. This process must stop since there are only finitely many U_i, so X has a closed point. ∎

Since every noetherian scheme is quasi-compact, every noetherian scheme has a closed point.

First we will construct a scheme without closed points using 3.3 and 2.7.

Theorem 4.2. *Let $A_n = (k(x_{n+1}, x_{n+2}, \ldots))[x_n]_{(x_n)}$. This is a DVR with generic point $P_{n-1} = Z_{n-1,n} = \operatorname{Spec} k(x_n, x_{n+1}, \ldots)$ and closed point $P_n = \operatorname{Spec} k(x_{n+1}, x_{n+2}, \ldots)$. Let $X_n = \operatorname{Spec} A_n$. Note that X_n's closed point is identified with X_{n+1}'s generic point so the $Z_{i,j}$ notation is justified. Let all other $Z_{i,j}$'s be the empty scheme. Let us denote the ringed space $\cup_{Z_{i,j}} X_n$ by X. Then X is a scheme.*

Proof: First notice that what we are doing is gluing DVR's end to end (closed point to open point) in infinite succession. What we have is a succession of points $P_0, P_1, P_2, \ldots$ with each P_i in the closure of all previous P's. Thus clearly X has no closed points. Thus the only open sets (besides the whole set) are finite sets of the form $Y_n = \{P_0, P_1, \ldots, P_n\}$. I claim these finite open sets are affine schemes. We will proceed by induction. Note that the inverse image of Y_n in all but the first n X_i's are empty (therefore the inverse image of the Y_n in higher $Z_{i,i+1}$'s are empty as well), so we can essentially ignore them. Further this means that the sections we get from them are only zero sections. A sort of minimal diagram of this situation is the following.

$$\begin{tikzcd} & Z_{1,2} \arrow[dl] \arrow[dr] & & Z_{2,3} \arrow[dl] \arrow[dr] & & Z_{3,4} \arrow[dl] \arrow[dr] & \\ X_1 & & X_2 & & X_3 & & \cdots \end{tikzcd}$$

Since these sets are all finite, we can get a perhaps clearer picture of what is going on by simply denoting them by their points.

$$\begin{tikzcd}[column sep=small]
& \{P_1\} \arrow[dl] \arrow[dr] & & \{P_2\} \arrow[dl] \arrow[dr] & & \{P_3\} \arrow[dl] \arrow[dr] & \\
\{P_0, P_1\} & & \{P_1, P_2\} & & \{P_2, P_3\} & & \dots
\end{tikzcd}$$

Obviously each pair of P_i's is identified in the coproduct giving the chain of points where each P_{i+1} is in the closure of P_i. Now consider $Y_1 = \{P_0, P_1\}$. This is clearly a scheme because it is the spectrum of $k(x_2, x_3, \ldots)[x_1] = A_1$. So suppose Y_i is a scheme for up to Y_n. Consider Y_{n+1}. By 2.7 we can construct Y_{n+1} by first gluing together Y_n and gluing on the rest. Our induction hypothesis tells us that Y_n is an affine scheme. Let us look at the setup from 2.7. The schemes $X' = Y_n$ and $X'' = X_{n+1}$ are both affine. The Z object from 2.7 is simply $Z_{n-1,n}$ since all the other possible $Z_{i,j}$'s are empty. But the map from Y_n to $Z_{n-1,n}$ is a closed immersion so we can apply 3.3 which tells us that $Y_n \cup_{Z_{n-1,n}} X_{n+1} = Y_{n+1}$ is a scheme. Therefore all Y_n's are schemes and since the collection of all such finite open Y_i's covers X, X is also a scheme. Note that X has no finite affine cover. ■

Now we will present an alternate view using valuation rings.

Let

$$A' = k[x_1, x_2, \ldots][\frac{x_1}{x_2}, \frac{x_1}{x_2^2} \ldots][\frac{x_2}{x_3}, \frac{x_2}{x_3^2} \ldots] \ldots$$

Now the principle of proposition 2.6 suggests that the local ring at the origin will be a local ring with no largest prime ideal among the non-maximal prime ideals. Let $A = A'_{(x_1, x_2, x_3, \ldots)}$ (of course the ideal $\mathfrak{m} = (x_1, x_2 \ldots)$ contains all the monomials of A'). A is the set of global sections of the scheme without closed points constructed above. Note that in A or A' I will call finite products of the generators of A (the $\frac{x_i}{x_{i+1}^n}$) monomials. If $n < m$ I will denote $\frac{x_n}{x_{n+1}} \frac{x_{n+1}}{x_{n+2}} \cdots \frac{x_{m-1}}{x_m^l}$ by $\frac{x_n}{x_m^l}$.

When we take the prime spectrum of A, we will get an infinite chain of points and then a closed point (corresponding to the maximal ideal). If we remove that closed point we get a scheme without closed points.

We are going to see that A is in fact a valuation ring with valuation to $G = \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$ which gives it properties very much like those of a DVR and actually makes the analysis surprisingly easy. The ordering we are going to use on G is the lexicographic order, $(n_1, n_2, \ldots) > (n'_1, n'_2, \ldots)$ if the first nonzero entry of $(n_1 - n'_1, n_2 - n'_2, \ldots)$ is greater than 0. Given $g \in G$ let $lv(g)$ be the value of the leading term of g, let $li(g)$ be the index of the leading term of g, and let g_i denote the i'th entry of g. So for example if $g = (0, 0, 4, -2, \ldots)$ then $lv(g) = 4$ while $li(g) = 3$ (the first two entries are zero). First we will define the valuation v on the monomials of A (or A'). Let m be a monomial of A, and view m as an element of Frac A, then we define $v(m)$ to be the degree (positive or negative) of x_i in m. So for example $v(x_1 \frac{x_1}{x_2^2} \frac{x_2}{x_3}) = (2, -1, -1, 0, 0, 0, \ldots)$. We shall now prove a number of quick results which will allow us to conclude that A is the desired ring.

Proposition 4.3. *For every $g \geq (0, 0, \ldots)$ there exists a unique monic monomial $x \in A$ (or A') such that $v(x) = g$.*

Proof: Let $l = li(g)$ and $n_l = lv(g)$. Note that without loss of generality we can assume $n_l = 1$ for if not we can find an x such that $v(x)$ matches g except at the leading term. That is $lv(v(x)) = 1$ and $v(xx_l^{lv(g)-1}) = g$. By the same method we can assume that $g_i \leq 0$ for all $i > l$ Let $t = \sum_{i>l} \mid g_i \mid$. Then let $x = \frac{x_l}{x_{l+1}^t} \prod_{i>l} \frac{x_{l+1}^{|g_i|}}{x_i^{|g_i|}}$ and note that it satisfies the desired condition. It is unique because monomials of Frac A satisfying that property are unique. ∎

We will now show that every element of A is a unit times a monomial but first we need a very important lemma.

Lemma 4.4. *If m_1 and m_2 are monomials of A and if $v(m_1) > v(m_2)$ then m_2 divides m_1.*

Proof: Let m_3 be the monomial corresponding to $v(m_1) - v(m_2)$, then $\lambda m_2 m_3 = m_1$ for some constant $\lambda \in k$ ∎

Proposition 4.5. *Every element $f \in A$ that is not a unit (so it has no constant term) is a unit multiplied by a unique monic monomial.*

Proof: Represent f as $(\lambda_1 m_1 + \ldots + \lambda_n m_n)/h$ where the $\lambda_i \in K$, $m_1 > m_2 > \ldots > m_n$ and h is an element of $A' - \mathfrak{m}$. Let $m_i' = \frac{m_i}{m_n} \in A' \subset A$. Thus $f = (\lambda_1 m_1' m_n + \ldots \lambda_n m_n)/h = m_n(\lambda_1 m_1' + \ldots \lambda_n)/h$ which is a monic monomial multiplied by a unit as desired. The fact that it is unique is easy to see since it is clear that any two distinct monomials are not associates (they do not differ by a unit). ∎

We can immediately conclude that every ideal of A is monomial. Now we have the required machinery to actually identify the prime ideals of A and use this to construct a scheme without closed points.

Theorem 4.6. *With A as described above,* $\operatorname{Spec} A - \mathfrak{m}$ *is a scheme without closed points.*

Proof: Suppose $P \in \operatorname{Spec} A$ and suppose $P \neq \mathfrak{m}$ and $P \neq (0)$. Note that $x_l \in P$ for some l since if $m \in P$ is any monomial then there exists l and n such that $v(x_l^n) > v(m)$ so that $x_l^n \in P$ which implies that $x_l \in P$. Let l be the largest number such that $x_l \in P$. Because $P \neq \mathfrak{m}$ such an l exists. Note that now for every monomial $m \in P$, $li(m) \leq l$, for if not we could choose l to be bigger. On the other hand, for all monomials $m \in A$ such that $li(m) = l$ it turns out that $m \in P$. To see this simply note that as in the proof of proposition 4.3 we can assume without loss of generality that $lv(m) = v(m)_l = 1$ and that $v(m)_i <= 0$ for all $i > l$. Thus we can represent m as $\frac{x_l}{m'}$ where m' is a monomial of $k[x_1, x_2 \ldots]$ and $li(m') > l$. Now $\frac{x_l}{m'} m' = x_l \in P$ so that $m' \in P$ or $\frac{x_l}{m'} \in P$ but m' cannot be in P because if it were it would contradict the maximality of l. Of course P automatically contains all monomials m such that $v(m) > v(x_l)$, which includes all monomials m such that $li(m) < l$. This completely identifies the monomials of P (which from this point forward we will denote by P_l). They are the monomials m such that $li(m) \leq l$ and since every ideal of A is monomial this completely determines P. At this point I have yet to prove that the P_i actually exist (as prime ideals), but if you look at A/P_i this is A without the first i variables of A (and all their quotients). It is easy to see that this leaves you with a ring isomorphic to A. Now note that P_i contains P_j for all $i > j$, thus the set of prime ideals of A is the set $\{(0), P_1, P_2, \ldots \mathfrak{m}\}$ and we also have $(0) \subset P_1 \subset P_2 \subset \ldots \subset \mathfrak{m}$. Therefore $\operatorname{Spec} A - \mathfrak{m}$ is a scheme without closed points.∎

Finally we should note that $\operatorname{Spec} A - \mathfrak{m}$ is the same scheme as the one constructed in 4.2. This is easy to see since the universal property 2.2 guarantees a map from the scheme constructed in 4.2 and it is not to difficult to see that this induces an isomorphism on the finite open (affine) subsets.

References

[1] Hartschorne, R., *Algebraic Geometry*, Springer-Verlag, New York (1977)
[2] Hartshorne, R., private correspondence with S. Kovács
[3] Saunders MacLane, *Categories for the Working Mathematician* 2nd edition, Springer-Verlag, New York (1998)

Department of Mathematics, University of Washington, Seattle, WA
E-mail address: schwede@math.washington.edu