THE ENUMERATIVE GEOMETRY OF RATIONAL AND ELLIPTIC CURVES IN PROJECTIVE SPACE

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ABSTRACT. We study the geometry of moduli spaces of genus 0 and 1 curves in $\mathbb{P}^n$ with specified contact with a hyperplane H. We compute intersection numbers on these spaces that correspond to the number of degree d curves incident to various general linear spaces, and tangent to H with various multiplicities along various general linear subspaces of H. (The numbers of classical interest, the numbers of curves incident to various general linear spaces and no specified contact with H, are a special case.) In the genus 0 case, these numbers are candidates for relative Gromov-Witten invariants of the pair $(\mathbb{P}^n, H)$, and in the genus 1 case they generalize the enumerative consequences of Kontsevich's reconstruction theorem for $\mathbb{P}^n$. The intersection numbers are rcursively computed by degenerating conditions. As an example, the enumerative geometry of quartic elliptic space curves is worked out in detail.

The methods used may be of independent interest, especially i) the surprisingly intricate geometry of maps of pointed curves to $\mathbb{P}^1$, and ii) the study of the space of curves in $\mathbb{P}^n$ via a smooth fibration (from an open set) to the space of maps of curves to $\mathbb{P}^1$. An unusual consequence of i) is an example of a map from a nodal curve to $\mathbb{P}^1$ that can be smoothed in two different ways.

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1. INTRODUCTION

In this paper, we study the geometry of moduli spaces of genus 0 and 1 curves in $\mathbb{P}^n$ with specified contact with a hyperplane H. Recursions are given to compute the number of degree d curves (of genus 0 or 1) incident to various general linear spaces, and tangent to H with various multiplicities along various general linear subspaces of H; call these numbers *enumerative invariants* of $\mathbb{P}^n$. (Gathmann refers to them as *degeneration invariants*, [Ga1].) The number usually of interest, the number of curves incident to various general linear spaces and with no specified contact with H, is a special case; call these *ordinary enumerative invariants*.

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Li and Ruan, and Ionel and Parker have proposed a definition of *relative Gromov-Witten invariants* ([LR], [R], [IPa]) in the symplectic category, which in the case of $(\mathbb{P}^n, H)$ agrees with the genus 0 enumerative invariants defined here. Enumerative invariants of $\mathbb{P}^n$ are thus potentially a good test-case for any candidate for an algebraic definition of relative Gromov-Witten invariants. Hence the recursions can be interpreted as a *full reconstruction theorem* for relative Gromov-Witten invariants of $\mathbb{P}^n$ (and genus 1 enumerative invariants, cf. Getzler's reconstruction theorem for genus 1 Gromov-Witten invariants of $\mathbb{P}^n$ [G]).

The approach to the enumerative problem is classical: one of the general linear spaces is specialized to lie in H, and the resulting degenerations and multiplicities are analyzed. (Even if one is only interested in ordinary enumerative invariants, one is forced to deal with all enumerative invariants.) Thanks to the power of Kontsevich's moduli space of maps, the overall strategy is simple, and most of the article is spent checking details. The reader may wish to see a few motivating examples to understand the issues that come up (Section 2; many phenomena are reminiscent of [CH]), and then read the basic definitions and strategy (Section 3).

As an example, many enumerative invariants of quartic elliptic space curves were computed (by hand, Section 8), including the fact that there are 52,832,040 such curves through 16 general lines in $\mathbb{P}^3$. This number was earlier computed by Avritzer and Vainsencher ([AVa], with the actual number corrected in [A]), and independently by Getzler (announced in [G], proof to appear in [GP]). Getzler's method gives recursions for ordinary enumerative invariants of genus 1 curves in $\mathbb{P}^3$.

Gathmann has extended many of these results to the case where H is replaced by a hypersurface ([Ga1]). He has also used extended these ideas to give a different algebro-geometric proof of the number of rational curves of all degrees on the quintic threefold ([Ga2]).

As a surprising aside, a map to $\mathbb{P}^1$ is given that has two distinct smoothings (Section 4.15).

1.1. *Brief history.* The enumerative geometry of space curves has been of interest since classical times (see [K2] for an excellent history; see also [KSX] and [PiZ]). Interest in such problems has been reinvigorated by recent ideas motivated by physics, and in particular Kontsevich's introduction of the moduli space of stable maps. Recursions for ordinary enumerative invariants of rational curves in $\mathbb{P}^n$ were one of the first applications of this space, via the First Reconstruction Theorem ([KoM], see also [RT]). Similar techniques have been brought to bear on (maps from) genus 1 curves (see [P], [I], [GP] for various enumerative results).

Caporaso and Harris used degeneration methods to give recursions for the enumerative geometry of plane curves (of arbitrary genus, [CH]). Although they use the Hilbert scheme, a reading of their paper from the perspective of stable maps is enlightening, and motivated this work. Such ideas can also be used to calculate genus g Gromov-Witten invariants of del Pezzo surfaces (or equivalently, count curves) and count curves on rational ruled surfaces ([V1]).

1.2. *Acknowledgements.* This article contains the majority of the author's 1997 Harvard Ph.D. thesis (and the e-print math.AG/9709007), and was partially supported by an NSERC 1967 Fellowship and a Sloan Foundation Dissertation Fellowship. The author is extremely grateful to his advisor, J. Harris, for inspiration and advice. Conversations with A.J. de Jong have greatly improved the exposition and argumentation. The author also wishes to thank D. Abramovich, M. Thaddeus, R. Pandharipande, T. Graber, T. Pantev, A. Vistoli, M. Roth, L. Caporaso, E. Getzler, and A. Gathmann for many useful discussions.

A. Gathmann has written a program computing genus 0 enumerative invariants of $\mathbb{P}^n$, available upon request from the authour.

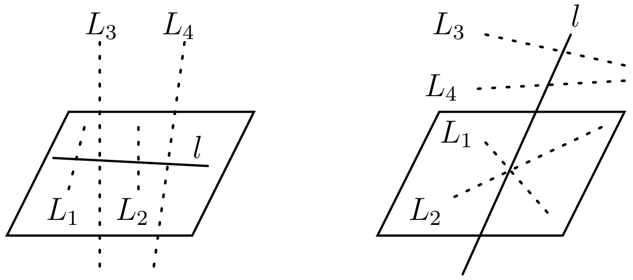

FIGURE 1. Possible positions of l after L_1 and L_2 have degenerated to H

2. MOTIVATING EXAMPLES

To count the number of curves in $\mathbb{P}^n$ incident to various general linear subspaces and tangent with various multiplicities to various general linear subspaces in H, we successively specialize the linear subspaces (not in H) to lie in H. By following through this idea in special cases, we get a preview of the behavior that will turn up in general.

2.1. Two lines through four fixed general lines in $\mathbb{P}^3$. Fix four general lines L_1, L_2, L_3, L_4 in $\mathbb{P}^3$, and a hyperplane H. There are a finite number of lines in $\mathbb{P}^3$ intersecting L_1, L_2, L_3, L_4. Call one of them l. We will specialize the lines L_1, L_2, L_3, and L_4 to lie in H one at a time and see what happens to l. First, specialize the line L_1 to (a general line in) H, and then do the same with L_2 (see Figure 1; H is represented by a parallelogram). If l doesn't pass through the intersection of L_1 and L_2, it must still intersect both L_1 and L_2, and thus lie in H. Then l is uniquely determined: it is the line through $L_3 \cap H$ and $L_4 \cap H$. Otherwise, if l passes through the point $L_1 \cap L_2$, it is once again uniquely determined (as only one line in $\mathbb{P}^3$ can pass through two general lines and one point — this can also be seen through further degeneration). This argument can be tightened to rigorously show the classical fact that there are two lines in $\mathbb{P}^3$ intersecting four general lines.

2.2. 92 conics through eight fixed general lines in $\mathbb{P}^3$. The example of conics in $\mathbb{P}^3$ is a simple extension of that of lines in $\mathbb{P}^3$, and gives a hint as to why stable maps are the correct way to think about these degenerations. Consider the question: How many conics pass through 8 general lines $L_1, \ldots, L_8$? (For another discussion of this classical problem, see [H] p. 26.) We introduce a pictorial shorthand that will allow us to easily follow the degenerations (see Figure 2).

We start with the set of conics through 8 general lines (the top row of the diagram — the label 92 indicates the number of such conics, which we will calculate last) and specialize one of the lines L_1 to H to get row 7. (The line L_1 in H is indicated by the dotted line in the figure.) When we specialize another line L_2, one of two things can happen: the conic can intersect H at the point $L_1 \cap L_2$ and one other (general) point, or it can intersect H once on L_1 and once on L_2 (at general points). (The requirement that the conic must pass through a fixed point in the first case is indicated by the thick dot in the figure.)

In this second case (the picture on the right in row 6), if we specialize another line L_3, one of three things can happen.

1. The conic can stay smooth, and not lie in H, in which case it must intersect H at $\{L_1 \cap L_3, L_2\}$ or $\{L_1, L_2 \cap L_3\}$ (hence the "$\times 2$" in the figure).
2. The conic could lie in H. In this case, there are eight conics through five fixed points $L_4 \cap H, \ldots, L_8 \cap H$ with marked points on the lines L_1, L_2, and L_3.

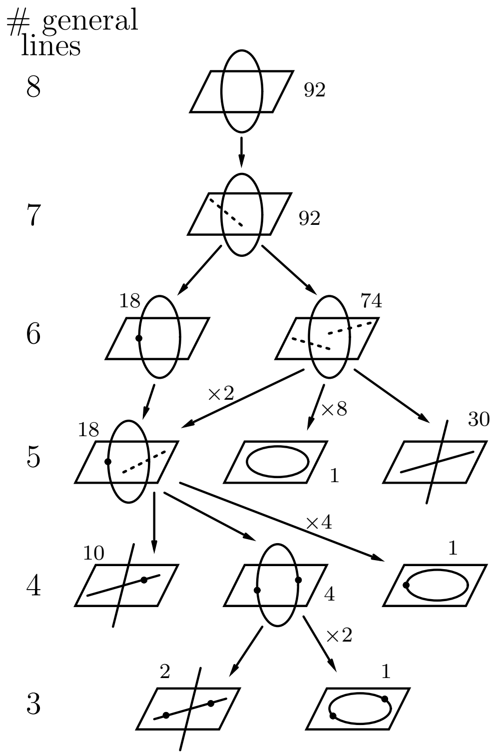

FIGURE 2. Counting 92 conics in $\mathbb{P}^3$ through 8 general lines

3. The conic can degenerate into the union of two intersecting lines, one (l_0) in H and one (l_1) not. These lines must intersect $L_4, \ldots, L_8$. (The line l_0 already intersects L_1, L_2, L_3, so these conditions are automatically satisfied.) Either three or four of $\{L_4, \ldots, L_8\}$ intersect l_1. In the first case, there are $\binom{5}{3}$ choices of the three lines, and two configurations (l_0, l_1) once the three lines are chosen (2 choices for l_1 from Section 2.1). In the second case there are a total of $\binom{5}{4} \times 2$ configurations by similar reasoning. Thus the total number of such configurations is 30.

We fill out the rest of the diagram in the same way. Then, using the enumerative geometry of lines in $\mathbb{P}^3$ and conics in $\mathbb{P}^2$ we can work our way up the table, attaching numbers to each picture, finally deducing that there are 92 conics through 8 general lines in $\mathbb{P}^3$. To make this argument rigorous, precise dimension counts and multiplicity calculations are needed.

The algorithm described in this article is slightly different: we parametrize rational curves with various conditions *and marked intersections with H*. In the case of conics through 8 lines, for example, we would count 184 conics through 8 lines with 2 marked points on H, and then divide by 2. The notation will then be cleaner. The resulting pictorial table is almost identical to Figure 2; the only difference is in the first two lines (see Figure 3).

2.3. Twisted cubics through 12 fixed general lines. The situation for curves in $\mathbb{P}^n$ in general is not much more complicated in principle than our calculations for conics in $\mathbb{P}^3$. Two additional twists come up, which are illustrated in the case of the $80,160$ twisted cubics through 12 general lines in $\mathbb{P}^3$, indicated pictorially in Figure 4. The third figure in row 8 represents a nodal (rational) cubic in H. There are 12 nodal cubics through 8 general points in $\mathbb{P}^2$. (The algorithm described in this paper will actually calculate $80,160 \times 3!$ cubics with marked points on H through 12 general lines.)

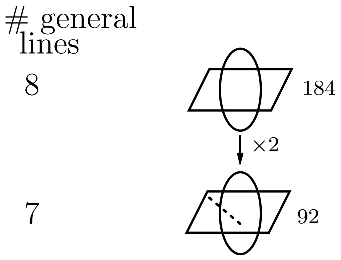

FIGURE 3. Counting 184 conics with two marked points on H through 8 general lines

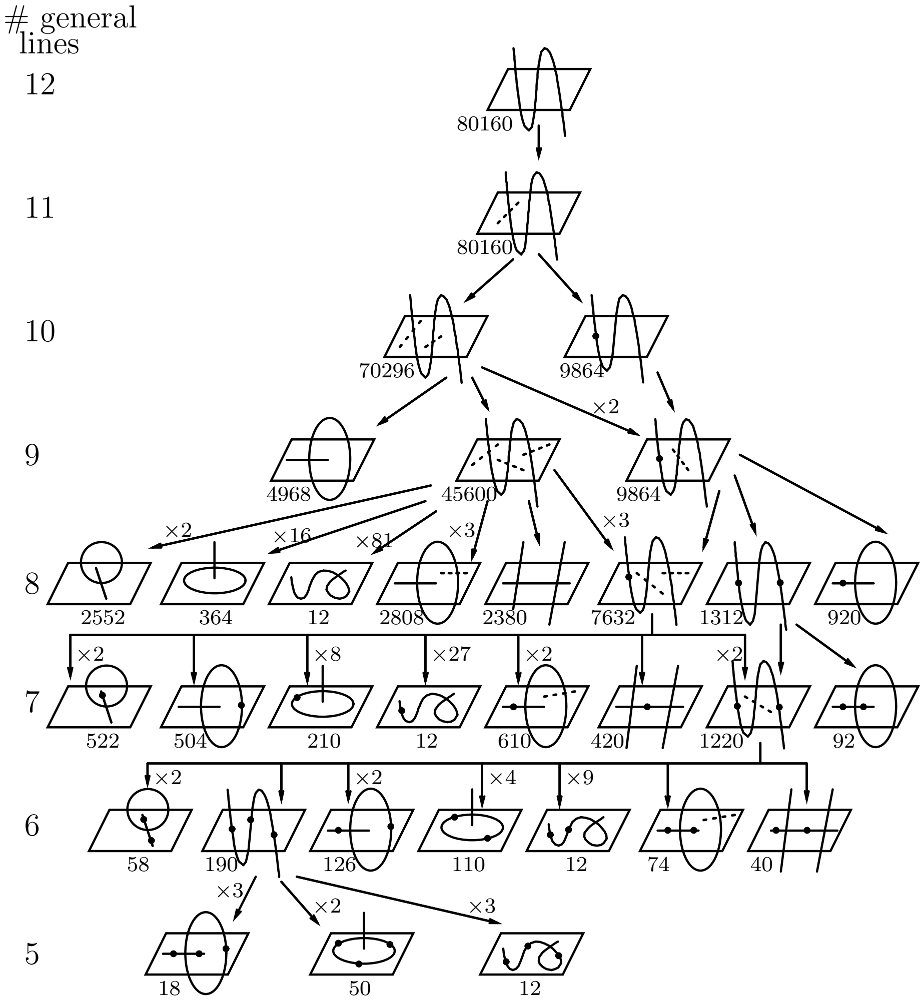

FIGURE 4. Counting 80, 160 cubics in $\mathbb{P}^3$ through 12 general lines

On the left side of row 8 we see a new degeneration (from twisted cubics through nine general lines intersecting H along three fixed general lines in H): a conic tangent to H, intersecting a line in H. (The tangency of the conic is indicated pictorially by drawing its lower horizontal tangent inside the parallelogram representing H.) We also have an unexpected multiplicity of 2 here.

The appearance of these new degenerations indicate why, in order to enumerate rational curves through general linear spaces by these degeneration methods, we must expand the set of curves under consideration to include those required to intersect H with given multiplicity, along linear subspaces.

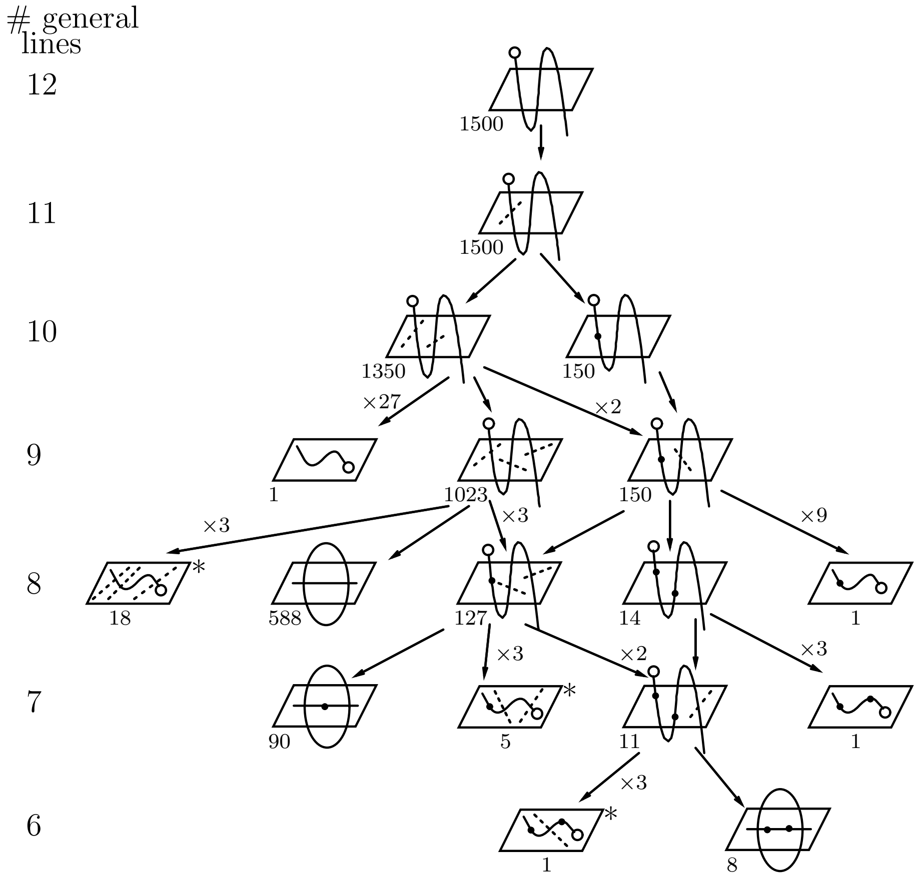

FIGURE 5. Counting 1500 elliptic cubics through 12 general lines in $\mathbb{P}^3$

2.4. Genus 1 cubics through 12 fixed general lines in space. The example of smooth elliptic cubics in $\mathbb{P}^3$ illustrates some of the degenerations we will see, and shows a new complication in genus 1. There are 1500 smooth elliptic cubics in $\mathbb{P}^3$ through 12 general lines, and we can use the same degeneration ideas to calculate this number. Figure 5 is a pictorial table of the degenerations; a smooth elliptic curve is represented by a squiggle with an open circle on the end.

The degenerations marked with an asterisk have a new twist. For example, consider the cubics through 9 general lines $L_1, \ldots, L_9$ and 3 lines L_{10}, L_{11}, L_{12} in H (the middle figure in row 9) and specialize L_9 to lie in H. The limit cubic could be a smooth plane curve in H (the left-most picture of row 8 in the figure). In this case, it must pass through the eight points $L_1 \cap H, \ldots, L_8 \cap H$. But there is an additional restriction. The cubics (before specialization) intersected L_{10}, L_{11}, L_{12} in three points p_{10}, p_{11}, p_{12} ($p_i \in L_i$), and as elliptic cubics are planar, these three points must have been collinear. Thus the possible limits are those curves in H through $L_1 \cap H, \ldots, L_8 \cap H$ and passing through collinear points p_{10}, p_{11}, p_{12} (with $p_i \in L_i$). (There is also a choice of a marked point of the curve on L_9, which will give a multiplicity of 3.) This collinearity condition can be written as $\pi^*(\mathcal{O}(1)) \cong \mathcal{O}(p_{10} + p_{11} + p_{12})$ in the Picard group of the curve.

We will have to count elliptic curves with such a divisorial condition involving the marked points; this locus forms a divisor on a family of stable maps. Fortunately, we can express this divisor in terms of divisors we understand well (Section 7.7). As a side benefit, we get enumerative data about elliptic curves in $\mathbb{P}^n$ with a divisorial condition as well.

3. Definitions and Strategy

3.1. Conventions. We work over $\mathbb{C}$. By *scheme*, we mean scheme of finite type over $\mathbb{C}$. By *variety*, we mean a separated integral scheme. *Curves* are assumed to be complete and reduced. All morphisms of schemes are assumed to be defined over $\mathbb{C}$, and fibre products are over $\mathbb{C}$ unless otherwise specified. If $f : C \to X$ is a morphism of schemes and Y is a closed subscheme of X, then define $f^{-1}(Y)$ as $C \times_X Y$; $f^{-1}Y$ is a closed subscheme of C. $\operatorname{Sing}(f)$ is the set of singular points of f (in C). Similar definitions apply for stacks, which are assumed to be of Deligne-Mumford type. A brief summary of basic facts about the moduli stack of stable maps $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^n, d)$ is included in Appendix A. If H is a hyperplane in $\mathbb{P}^n$, and q is a marked point, we will occasionally let $\{\pi(q) \in H\}$ denote the Cartier divisor $ev_q^* H$ in order to be geometrically suggestive. If $\mathcal{X}$ is a zero-dimensional scheme (or stack), let $\#\mathcal{X}$ be the number of points of $\mathcal{X}$. (One should really count each point with multiplicity $1/G$, where G is the cardinality of the isotopy group of the point, but this will be 1 in these applications.)

If Δ is a collection (of subspaces of $\mathbb{P}^n$, for example), indexed by a set $S(\Delta)$, let $|\Delta|$ be the cardinality of $S(\Delta)$. We use set notation for collections.

Throughout this paper, H is a hyperplane of $\mathbb{P}^n$, and A is a hyperplane of H.

3.2. The stacks $\mathcal{X}$ and $\mathcal{W}$. Fix positive integers n and d. Suppose $\Delta = \{\Delta^\alpha\}_{\alpha \in S(\Delta)}$ is a collection of general linear spaces of $\mathbb{P}^n$ indexed by a set $S(\Delta)$. (We use a collection rather than a set as we will need to consider the case when all of the Δ^α are dimension n, i.e. $\mathbb{P}^n$.) For each positive integer m, suppose $\Gamma_m = \{\Gamma_m^\alpha\}_{\alpha \in S(\Gamma_m)}$ is a collection of general linear spaces of H indexed by a set $S(\Gamma_m)$, where $\sum_m m|\Gamma_m| = d$. Let $\Gamma = \{\Gamma_m\}_{m>0}$. Then define $\mathcal{X}_n(d, \Gamma, \Delta)$ to be the (stack-theoretic) closure in $\overline{\mathcal{M}}_{0,\sum|\Gamma_m|+|\Delta|}(\mathbb{P}^n, d)$ of points corresponding to stable maps $\pi : C \to \mathbb{P}^n$ of genus 0 curves with marked points q^α ($\alpha \in S(\Delta)$) and p_m^α ($m > 0$, $\alpha \in S(\Gamma_m)$), such that

(i) $\pi(q^\alpha) \in \Delta^\alpha$ for all $\alpha \in S(\Delta)$, and $\pi(p_m^\alpha) \in \Gamma_m^\alpha$ for all $m > 0$, $\alpha \in S(\Gamma_m)$.
(ii) $\pi^* H = \sum_{m, \alpha \in S(\Gamma_m)} m p_m^\alpha$.

(As a consequence, no component of C is mapped to H.) Informally, this stack parametrizes rational curves incident (at a marked point) to the linear spaces $\{\Delta^\alpha\}$, and m-fold tangent to H (at a marked point) along the space Γ_m^α (for all $m > 0$, $\alpha \in S(\Gamma_m)$).

Define $\mathcal{W}_n(d, \Gamma, \Delta)$ to be the closure in $\overline{\mathcal{M}}_{1,\sum|\Gamma_m|+|\Delta|}(\mathbb{P}^n, d)$ of points corresponding to stable maps $\pi : C \to \mathbb{P}^n$ of genus 1 curves with marked points q^α ($\alpha \in S(\Delta)$) and p_m^α (for all $m > 0$, $\alpha \in S(\Gamma_m)$), satisfying (i) and (ii) above and also

(iii) No connected union of components of C of arithmetic genus 1 is contracted.

Informally, this stack parametrizes genus 1 curves incident to the linear spaces $\{\Delta_\alpha\}$, and m-fold tangent to H along the space Γ_m^α (for all $m > 0$, $\alpha \in S(\Gamma_m)$).

The subscript n will often be suppressed to keep the notation from becoming too complicated.

3.3. The Strategy. We calculate $\#\mathcal{W}_n(d, \Gamma, \Delta)$ (or $\#\mathcal{X}_n(d, \Gamma, \Delta)$) by degenerating one of the linear spaces Δ^β (general in $\mathbb{P}^n$) to a general linear space in H, and observing how the maps corresponding to points of $\mathcal{W}_n(d, \Gamma, \Delta)$ degenerate. We can express this degeneration method as follows. Let Δ' be the same as Δ except Δ'^β is dimension one greater than Δ^β. Let D_H be the Cartier divisor $ev_\beta^* H$ (i.e. $\pi(q^\beta) \in H$), and $D_{H'}$ the Cartier divisor $ev_\beta^* H'$, where H' is a general hyperplane in $\mathbb{P}^n$. Then $\mathcal{W}_n(d, \Gamma, \Delta) = \mathcal{W}_n(d, \Gamma, \Delta') \cap D_{H'}$ (Proposition 5.3(c)). The degeneration corresponds to counting the points of $\mathcal{W}_n(d, \Gamma, \Delta') \cap D_H$, with multiplicity. As D_H is linearly equivalent to $D_{H'}$, these numbers are the same.

In short, we must understand the irreducible components (and the corresponding multiplicities) of the divisor $D_H = \sum m_i D_i$ on $\mathcal{W}_n(d, \Gamma, \Delta)$. This is the key problem addressed in this paper. We make three reductions to simplify the problem.

Reduction A. The condition of requiring a marked point to lie on a fixed general hyperplane imposes one transverse condition on any irreducible substack of $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^n, d)$ (by Kleiman-Bertini 5.1), so requiring a marked point to lie on a codimension k fixed general linear space imposes k transverse conditions. Hence it suffices to know the components and multiplicities of D_H when each Γ^α_m is H and each Δ^α is $\mathbb{P}^n$ (as one can then reduce to the case when the Γ^α_m and Δ^α are smaller by intersecting the D_i's with the appropriate transverse conditions).

Reduction B. Now that there are no conditions on the marked points q^α, it suffices to know the components and multiplicities of D_H in the case when $S(\Delta) = \{\beta\}$ (as we can then reduce to the case where $|\Delta| > 1$ by adding marked points).

Reduction C. Projection from the hyperplane A of H gives a rational map $\mathbb{P}^n \dashrightarrow \mathbb{P}^1$ (sending H to a point $\infty \in \mathbb{P}^1$) that induces a rational map $\rho_A : \overline{\mathcal{M}}_{g,m}(\mathbb{P}^n, d) \dashrightarrow \overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, d)$ that is a morphism on an open set corresponding to maps $\pi : C \to \mathbb{P}^n$ where $\pi^{-1}A = \emptyset$. It extends to a morphism on the open set $\mathcal{V}$ where $\pi^{-1}A$ is a union of reduced smooth points of C, and ρ_A is smooth on $\mathcal{V}$ (Proposition 5.5). Each component of D_H meets $\mathcal{V}$. Hence if we can solve the problem for $n = 1$, we can solve the problem in general: the divisor D_H (restricted to $\mathcal{V}$) is the pullback of the corresponding divisor D_∞ on $\mathcal{W}_1(d, \Gamma', \Delta' = \{\mathbb{P}^1\})$, and the components of D_H (restricted to $\mathcal{V}$) are pullbacks of the analogous components of D_∞ (and also with $\mathcal{W}$ replaced by $\mathcal{X}$). As ρ_A is smooth on $\mathcal{V}$, the multiplicities are the same.

For this reason, we first turn to the space of maps from curves to $\mathbb{P}^1$ with specified ramification (at marked points) over a point $\infty \in \mathbb{P}^1$, with one other marked point q^β, and find the components and multiplicities of $D_\infty = ev^*_{q^\beta}\infty = \{\pi(q^\beta) = \infty\}$ (Section 4). This is technically the most intricate part of the argument.

3.4. *Summary.* In short, the proof of the recursion for the number of genus 0 and 1 curves with prescribed incidences and tangencies is as follows. We first study stacks of maps to $\mathbb{P}^1$, with prescribed ramification over ∞ (at marked points), and one other marked point q^β, and find the components and multiplicities of $ev^*_\beta\infty$ (Section 4). Then, pulling back by the smooth morphism ρ_A (Reduction C), we have a result on stacks of maps to $\mathbb{P}^n$ with prescribed intersection with a hyperplane H and one other marked point q^β, giving components and multiplicities of $ev^*_\beta H$. By adding additional marked points (Reduction B) and requiring them to lie on various numbers of general hyperplanes (Reduction A), we have a linear equivalence of divisors on stacks of maps to $\mathbb{P}^n$ with various incidence and tangency conditions (Section 6). If the stack is one-dimensional, we get an expression giving each enumerative invariant in terms of "simpler" enumerative invariants, i.e. recursions (Section 7.1).

3.5. Definitions: Components of D_H. Suppose $\mathcal{A}$ is a stack of m-pointed, arithmetic genus g, degree d stable maps to $\mathbb{P}^n$, and let $D_i = ev^*_i H$ ($1 \le i \le m$). If $\prod_{i=1}^m D_i^{n_i}[\mathcal{A}] = 0$ for all m-tuples $(n_1, \ldots, n_m)$ adding to $\dim \mathcal{A}$, we say $\mathcal{A}$ is *enumeratively irrelevant*; otherwise it is *enumeratively relevant*. The components of D_H that are enumeratively irrelevant will not contribute to enumerative calculations, and may be discarded.

If $\mathcal{A}^{(j)} := \mathcal{A} \times_{\overline{\mathcal{M}}_{g,m}(\mathbb{P}^n,d)} \overline{\mathcal{M}}_{g,m+j}(\mathbb{P}^n, d)$ is *enumeratively irrelevant* for all $j \ge 0$, we say $\mathcal{A}$ is *stably enumeratively irrelevant*. (The stack $\mathcal{A}^{(1)}$ is the universal curve over $\mathcal{A}$, and $\mathcal{A}^{(j+1)}$ is the universal curve over $\mathcal{A}^{(j)}$.) Informally speaking, if the "image of $\mathcal{A}$ in the Chow variety" is of dimension less than that of $\mathcal{A}$, then $\mathcal{A}$ is stably enumeratively irrelevant. (See [V2] 2.1

for a similar definition.) Enumerative irrelevance can arise because of moduli in a contracted component, see Proposition 5.2.

3.6. The components of the divisor D_H on stacks of form $\mathcal{X}$ or $\mathcal{W}$ that are enumeratively relevant will turn out to be stacks of the form defined below. The enumerative geometry of these will be obviously related to the enumerative geometry of stacks of the form $\mathcal{X}$ and $\mathcal{W}$.

Fix n, d, Γ, Δ, and a non-negative integer l. Let $\sum_{k=0}^{l} d(k)$ be a partition of d. Let the points $\{p_m^\alpha\}_{m,\alpha}$ be partitioned into $l+1$ subsets $\{p_m^\alpha(k)\}_{m,\alpha}$ for $k = 0, \dots, l$. This induces a partition of each Γ_m into $\coprod_{k=0}^{l} \Gamma_m(k)$. Let the points $\{q^\alpha\}_\alpha$ be partitioned into $l+1$ subsets $\{q^\alpha(k)\}_\alpha$ for $k = 0, \dots, l$. This induces a partition of Δ into $\coprod_{k=0}^{l} \Delta(k)$. For $k > 0$, define $m^k := d(k) - \sum_m m|\Gamma_m(k)|$, and assume $m^k > 0$ for all $k = 1, \dots, l$. In Definitions 3.7, 3.8, 3.10, and 3.11 below, the marked points on $\overline{\mathcal{M}}_{g,\sum|\Gamma_m|+|\Delta|}(\mathbb{P}^n, d)$ are labelled $\{p_m^\alpha\}_{\alpha\in S(\Gamma)}$ and $\{q^\alpha\}_{\alpha\in S(\Delta)}$

Substacks of the following form will appear as components of D_H on stacks of the form $\mathcal{X}(d,\Gamma,\Delta)$.

3.7. *Definition.* The stack

$$\mathcal{Y}_n(d(0),\Gamma(0),\Delta(0);\dots;d(l),\Gamma(l),\Delta(l))$$

is the (stack-theoretic) closure of the locally closed substack of $\overline{\mathcal{M}}_{0,\sum|\Gamma_m|+|\Delta|}(\mathbb{P}^n, d)$ representing stable maps $(C, \{p_m^\alpha\}, \{q^\alpha\}, \pi)$ satisfying the following conditions

- Y1. The curve C consists of $l+1$ irreducible components $C(0), \dots, C(l)$ with all components meeting $C(0)$. The map π has degree $d(k)$ on curve $C(k)$ $(0 \le k \le l)$.
- Y2. The points $\{p_m^\alpha(k)\}_{m,\alpha}$ and $\{q^\alpha(k)\}_\alpha$ lie on $C(k)$, and $\pi(p_m^\alpha(k)) \in \Gamma_m^\alpha(k)$, $\pi(q^\alpha(k)) \in \Delta^\alpha(k)$.
- Y3. As sets, $\pi^{-1}H = C(0) \cup \{p_m^\alpha\}_{m,\alpha}$, and for $k > 0$,

$$(\pi\mid_{C(k)})^* H = \sum_{m,\alpha} m p_m^\alpha(k) + m^k(C(0)\cap C(k)).$$

Pictorial representations of such maps are given in the Figures of Section 2. Note that $d(k) > 0$ for all positive k by the last condition.

Substacks of the following forms will appear as components of D_H on stacks of the form $\mathcal{W}(d,\Gamma,\Delta)$.

3.8. *Definition.* The stack

$$\mathcal{Y}_n^a(d(0),\Gamma(0),\Delta(0);\dots;d(l),\Gamma(l),\Delta(l))$$

is the closure of the locally closed substack of $\overline{\mathcal{M}}_{1,\sum|\Gamma_m|+|\Delta|}(\mathbb{P}^n, d)$ representing stable maps $(C, \{p_m^\alpha\}, \{q^\alpha\}, \pi)$ satisfying conditions Y1–Y3 above, and

- Y^{a}4. The curve $C(1)$ has genus 1 (and the other components are genus 0).

3.9. *Remark.* For given d, Γ, Δ, m, and choices $\alpha \in \Gamma_m$, $\beta \in \Gamma$, let Γ' be the same as Γ except $\Gamma'^\alpha_m = \Gamma_m^\alpha \cap \Delta^\beta$. Then there is a natural isomorphism

$$\phi : \mathcal{X}_n(d,\Gamma',\Delta\setminus\{\Delta^\beta\}) \to \mathcal{Y}_n(0,\{\Gamma_m^\alpha\},\{\Delta^\beta\};d,\Gamma\setminus\{\Gamma_m^\alpha\},\Delta\setminus\{\Delta^\beta\}).$$

The map from the left to the right involves gluing a contracted $\mathbb{P}^1$ (with marked points p_m^α and q^β) to the point p'^α_m.

Similarly, there is a natural isomorphism (which we sloppily denote ϕ as well)

$$\phi : \mathcal{W}_n(d,\Gamma',\Delta\setminus\{\Delta^\beta\}) \to \mathcal{Y}_n^a(0,\{\Gamma_m^\alpha\},\{\Delta^\beta\};d,\Gamma\setminus\{\Gamma_m^\alpha\},\Delta\setminus\{\Delta^\beta\}).$$

3.10. *Definition.* The stack

$$\mathcal{Y}^b_n(d(0), \Gamma(0), \Delta(0); \dots ; d(l), \Gamma(l), \Delta(l))$$

is the closure of the locally closed substack of $\overline{\mathcal{M}}_{1,\sum|\Gamma_m|+|\Delta|}(\mathbb{P}^n, d)$ representing stable maps $(C, \{p^\alpha_m\}, \{q^\alpha\}, \pi)$ satisfying conditions Y1–Y2 above, and

Y$^{\mathrm{b}}$3. As sets, $\pi^{-1}H = C(0) \cup \{p^\alpha_m\}_{m,\alpha}$, and for $k > 1$,

$$(\pi \mid_{C(k)})^* H = \sum_{m,\alpha} m p^\alpha_m(k) + m^k(C(0) \cap C(k)).$$

Y$^{\mathrm{b}}$4. All components of C are rational. The curves $C(0)$ and $C(1)$ intersect at two distinct points $\{a_1, a_2\}$. (These points are not marked; monodromy may exchange them.) Also,

$$(\pi \mid_{C(1)})^* H = \sum_{m,\alpha} m p^\alpha_m(k) + m^1_1 a_1 + m^1_2 a_2$$

where $m^1_1 + m^1_2 = m^1$.

Thus $\mathcal{Y}^b(d(0), \dots, \Delta(l))$ is naturally the union of $[m^1/2]$ (possibly reducible) stacks (where $[\cdot]$ is the greatest-integer function), indexed by m^1_1. For convenience, label these stacks

$$\{\mathcal{Y}^b(d(0), \dots, \Delta(l))_{m^1_1}\}_{1 \le m^1_1 < m^1},$$

so $\mathcal{Y}^b(d(0), \dots, \Delta(l))_{m^1_1} = \mathcal{Y}^b(d(0), \dots, \Delta(l))_{m^1 - m^1_1}$.

3.11. *Definition.* The stack

$$\mathcal{Y}^c_n(d(0), \Gamma(0), \Delta(0); \dots ; d(l), \Gamma(l), \Delta(l))$$

is the closure of the locally closed substack of $\overline{\mathcal{M}}_{1,\sum|\Gamma_m|+|\Delta|}(\mathbb{P}^n, d)$ representing stable maps $(C, \{p^\alpha_m\}, \{q^\alpha\}, \pi)$ satisfying conditions Y1–Y3 above, and

Y$^{\mathrm{c}}$4. The curve $C(0)$ has genus 1 (and the other components are genus 0). The morphism π has positive degree on every component.

Y$^{\mathrm{c}}$5. In $\operatorname{Pic}(C(0))$,

$$\pi^*(\mathcal{O}_{\mathbb{P}^n}(1)) \otimes \mathcal{O}_{C(0)} \left(\sum_{k=1}^{l} m^k (C(0) \cap C(k)) \right) \cong \mathcal{O}_{C(0)} \left(\sum_{m,\alpha \in \Gamma_m(0)} m p^\alpha_m(0) \right).$$

3.12. *Remark.* This Picard condition Y$^{\mathrm{c}}$5 was actually present in $\mathcal{Y}$, $\mathcal{Y}^a$ and $\mathcal{Y}^b$, but as $C(0)$ was rational in each of these cases, the requirement reduced to

$$d(0) + \sum_{k=1}^{l} m^k = \sum_m m|\Gamma_m(0)|$$

which was always true.

The six types of stacks $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Y}^a$, $\mathcal{Y}^b$, $\mathcal{Y}^c$ are illustrated in Figure 6. In the figure, the dual graph of the curve corresponding to a general point of the stack is given. Vertices corresponding to components mapped to H are labelled with an H, and vertices corresponding to genus 1 components are open circles.

Because of the divisorial condition Y$^{\mathrm{c}}$5 in the definition of $\mathcal{Y}^c$, we will also be interested in the variety parametrizing smooth degree d genus 1 curves in $\mathbb{P}^n$ ($n \ge 2$) with a condition in the Picard group of the curve involving the marked points and $\pi^*(\mathcal{O}_{\mathbb{P}^n}(1))$. Let $\Delta = \{\Delta^\alpha\}_{\alpha \in S(\Delta)}$ be a set of linear spaces in $\mathbb{P}^n$. Let $\mathcal{D}$ be a linear combination of the formal variables $\{q^\alpha\}_{\alpha \in S(\Delta)}$ with integral coefficients summing to d.

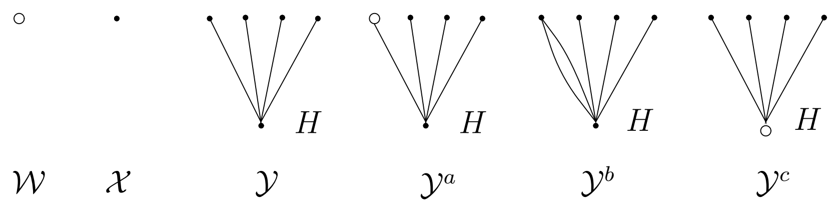

FIGURE 6. Six types of stacks

3.13. *Definition.* The stack $\mathcal{Z}_n(d, \Delta)_{\mathcal{D}}$ is the (stack-theoretic) closure of the locally closed subset of $\overline{\mathcal{M}}_{1,|\Delta|}(\mathbb{P}^n, d)$ (where the points are labelled $\{q^\alpha\}_\alpha$) representing stable maps $(C, \{q^\alpha\}, \pi)$ where C is smooth, $\pi(q^\alpha) \in \Delta^\alpha$ for all α, and $\pi^*(\mathcal{O}_{\mathbb{P}^n}(1)) \cong \mathcal{O}_C(\mathcal{D})$ in $\operatorname{Pic}(C)$.

For example,

$$\mathcal{Z}_2(d = 4, \Delta = \{11 \text{ general points}\})_{q^1+q^2+q^3+q^4}.$$

parametrizes the finite number of two-nodal quartic plane curves C through 11 fixed general points $\{q^j\}_{1 \leq j \leq 11}$ such that $q^1 + q^2 + q^3 + q^4$ is linearly equivalent to the hyperplane section on the normalization of the curve C. (There are 62 such curves, see the example at the end of Section 7.7 and Table 2.)

When Γ and Δ are general, all of the varieties $\mathcal{W}$, $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Y}^a$, $\mathcal{Y}^b$, $\mathcal{Y}^c$, $\mathcal{Z}$ defined above will be seen to be of the expected dimension (Proposition 5.7).

4. MAPS TO $\mathbb{P}^1$

4.1. *Quasi-stable maps.* For the purposes of this Section, define a quasi-stable map in the same way as a stable map, except the source curve is not required to be connected. Then the entire theory of stable maps carries through for quasi-stable maps, and for smooth projective X there is a fine moduli stack, which we'll denote $\overline{\mathcal{M}}_{g,m}(X, \beta)^Q$, of finite type, and $\overline{\mathcal{M}}_{g,m}(X, \beta)$ is an open and closed substack of $\overline{\mathcal{M}}_{g,m}(X, \beta)^Q$. For more details on this (essentially trivial) variation on stable maps, see [V1] Section 2.5.

4.2. Deformations of maps from curves to curves. Suppose $\pi : C \to \mathbb{P}^1$ is a stable degree d map with marked points $\{p_i\}$. Call étale neighborhoods of connected components A of $\operatorname{Sing}(\pi) \cup \{p_i\} \subset C$ (containing only one copy of A and no other points of $\operatorname{Sing}(\pi) \cup \{p_i\}$) *special loci* of π. Special loci are étale neighborhoods of ramification points of C, nodes of C, marked points, or unions of contracted components of π. The map π is stable, so the functor parametrizing deformations of π is pro-representable by the formal neighborhood Def of the corresponding point in the moduli stack of stable maps.

If $A_1, \dots, A_n$ are the special loci of π (or more precisely, if étale neighborhoods of $A_1, \dots, A_n$ are the special loci), let $\operatorname{Def}_{A_1}, \dots, \operatorname{Def}_{A_n}$ be the deformation spaces (i.e. hulls) of the special loci. (Implicit here is the fact that Def_{A_j} is independent of the neighborhood of A_j chosen; this will follow from the proof of the following proposition.)

4.3. *Proposition. — The natural map* $\operatorname{Def} \to \operatorname{Def}_{A_1} \times \cdots \times \operatorname{Def}_{A_n}$ *is an isomorphism.*

In the analytic category, this proposition is clear.

Proof. The deformation theory of π is controlled by $\operatorname{Ext}^{\cdot}(\underline{\Omega}_\pi, \mathcal{O}_C)$ (see Appendix A). Let $e_j : C_j \to C$ be an étale neighborhood of A_j as described above, so Def_{A_j} is constructed using $\operatorname{Ext}(e_j^* \underline{\Omega}_\pi, e_j^* \mathcal{O}_C)$ (as $e_j^* \underline{\Omega}_\pi = \underline{\Omega}_{\pi \circ e_j}$).

Let $K = \operatorname{Ker}(\underline{\Omega}_\pi)$ and $Q = \operatorname{Coker}(\underline{\Omega}_\pi)$, so K is supported on the special loci and hence splits canonically into $\oplus_j K_j$, with K_j supported on A_j (and similarly for $Q = \oplus_j Q_j$).

There are exact sequences

$$0 \to K[1] \to \underline{\Omega}_\pi \to Q \to 0 \tag{1}$$

and (for each j)

$$0 \to K_j[1] \to e_j^* \underline{\Omega}_\pi \to Q_j \to 0 \tag{1_j}$$

with morphisms (1) $\to$ (1_j) induced by e_j. By considering (1) $\to \oplus_j$ (1_j) and taking the associated long exact Ext sequence, we have

$$\begin{array}{ccccccccc} \dots & \to & \operatorname{Ext}^i(Q, \mathcal{O}_C) & \to & \operatorname{Ext}^i(\underline{\Omega}_\pi, \mathcal{O}_C) & \to & \operatorname{Ext}^{i-1}(K, \mathcal{O}_C) & \to & \dots \\ & & || & & \downarrow & & || & & \\ \dots & \to & \oplus_j \operatorname{Ext}^i(Q_j, \mathcal{O}_{C_j}) & \to & \oplus_j \operatorname{Ext}^i(e_j^* \underline{\Omega}_\pi, \mathcal{O}_{C_j}) & \to & \oplus_j \operatorname{Ext}^{i-1}(K_j, \mathcal{O}_{C_j}) & \to & \dots . \end{array} \tag{2}$$

By the five lemma, the vertical arrow is an isomorphism for all i, and by the construction of Def from $\operatorname{Ext}^i(\underline{\Omega}_\pi, \mathcal{O}_C)$ (and Def_{A_j} from $\operatorname{Ext}^i(\underline{\Omega}_{\pi \circ e_j}, \mathcal{O}_{C_j})$), the result follows. $\square$

4.4. Substacks of $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, d)$. Fix a positive integer d and a point ∞ on $\mathbb{P}^1$, and let $\vec{h} = (h_1, h_2, \dots)$ represent a partition of d with h_1 1's, h_2 2's, etc., so $\sum_m m h_m = d$. Let $\mathcal{V} = \mathcal{V}^{d,g}(\vec{h})$ be the closure in $\overline{\mathcal{M}}_{g, \sum h_m + 1}(\mathbb{P}^1, d)^Q$ of points representing *quasi-stable* maps $(C, \{p_m^\alpha\}, q, \pi)$ where C is a smooth curve with $\sum h_m + 1$ (distinct) marked points $\Gamma_m = \{p_m^\alpha\}_{1 \leq \alpha \leq h_m}$ and q, and $\pi^*(\infty) = \sum_{m,\alpha} m p_m^\alpha$. (For example, $\mathcal{X}_1(d, \Gamma, \Delta)$ is an open and closed substack of $\mathcal{V}^{d,0}(\vec{h})$, where Γ_m consists of h_m copies of ∞, and $\Delta = \{\mathbb{P}^1\}$. Similarly, $\mathcal{W}_1(d, \Gamma, \Delta)$ is an open and closed subset of $\mathcal{V}^{d,1}(\vec{h})$.)

For the map corresponding to a general point in $\mathcal{V}$, each special locus A_j is either a marked ramification above the point ∞, a simple unmarked ramification (of which there are $d + 2g - 2 + \sum h_m$ by Riemann-Hurwitz), or the point q (at which π is smooth). In these three cases, the formal deformation space of A_j inside $\mathcal{V}$ is 0, $\operatorname{Spf} \mathbb{C}[[t]]$, and $\operatorname{Spf} \mathbb{C}[[t]]$ respectively. Thus

$$\dim \mathcal{V} = d + 2g - 1 + \sum h_m \tag{3}$$

Let D_∞ be the divisor $ev_q^*(\infty) = \{\pi(q) = \infty\}$. There are three natural questions to ask:

1. What are the components of the divisor D_∞?
2. With what multiplicity do they appear?
3. What is the local structure of $\mathcal{V}$ near these components?

We partially answer these three questions.

Fix a component $\mathcal{Y}$ of the divisor D_∞ and a map $(C, \{p_m^\alpha\}, q, \pi)$ corresponding to the general element of $\mathcal{Y}$. Notice that π collapses a component of C to ∞, as otherwise $\pi^{-1}(\infty)$ is a union of points, and

$$d = \deg \pi^*(\infty) \geq \deg_q \pi^*(\infty) + \sum_{m,\alpha} \deg_{p_m^\alpha} \pi^*(\infty) \geq 1 + \sum_m m h_m = d + 1.$$

Let $C(0)$ be the connected component of $\pi^{-1}(\infty)$ containing q, and let $\tilde{C}$ be the closure of $C \setminus C(0)$ in C (see Figure 7; $C(0)$ is the union of those curves contained in the dotted rectangle, and $\tilde{C}$ is the rest of C).

Let $\Gamma_m(0) = \{p_m^\alpha(0)\}$ be the points of Γ_m on $C(0)$, and $\tilde{\Gamma}_m = \{\tilde{p}_m^\alpha\} = \Gamma_m \setminus \Gamma_m(0)$ be the points on $\tilde{C}$ (with $h_m(0) = |\Gamma_m(0)|$, $\tilde{h}_m = |\tilde{\Gamma}_m|$). Let s be the number of intersections of $C(0)$ and $\tilde{C}$, and label these points $r^1, \dots, r^s$. Thus $g = p_a(C(0)) + p_a(\tilde{C}) + s - 1$. Let m^k be the multiplicity of $(\pi|_{\tilde{C}})^*(\infty)$ at r^k. The data $(m^1, \dots, m^s)$ is constant for any choice of $(C, \{p_m^\alpha\}, q, \pi)$ in an open subset of $\mathcal{Y}$.

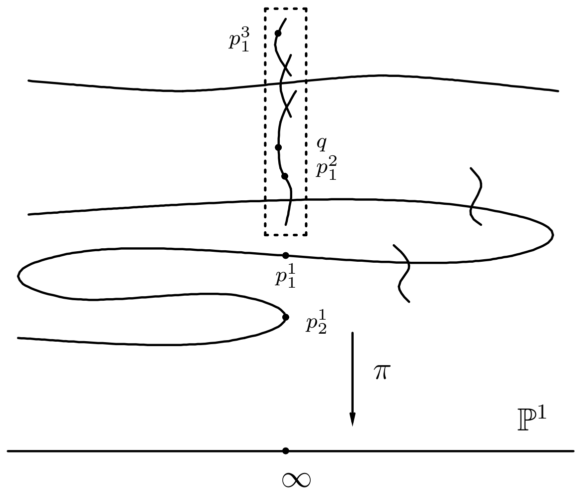

FIGURE 7. The map $(C, \{p_m^\alpha\}, q, \pi\}) \in \mathcal{Y}$

4.5. *Proposition. — The map* $(\tilde{C}, \{p_m^\alpha(0)\}, \{r^k\}, \pi)$ *has no collapsed components, and only simple ramification away from* $\pi^{-1}(\infty)$. *The curve* $\tilde{C}$ *is smooth.*

The map $(\tilde{C}, \{p_m^\alpha(0)\}, \{r^k\}, \pi)$ will turn out to correspond to a general element in $\mathcal{V}^{d,g'}(\vec{h'})$ for some g', $\vec{h'}$.

Proof. Let $A_1, \ldots, A_l$ be the special loci of π, and say $q \in A_1$.

The map $(C, \{p_m^\alpha\}, q, \pi)$ lies in $\mathcal{V}$ and hence can be deformed to a curve where each special locus is either a marked ramification above ∞, a simple unmarked ramification, or the point q. If A_k $(k > 1)$ is not one of these three forms then by Proposition 4.3 there is a deformation of the map $(C, \{p_m^\alpha\}, q, \pi)$ preserving π at A_i $(i \neq k)$ but changing A_k into a combination of special loci of these three forms. Such a deformation (in which A_1 is preserved and thus still smoothable) is actually a deformation in the divisor $D_\infty = ev_q^* \infty = \{\pi(q) = \infty\}$, contradicting the generality of $(C, \{p_m^\alpha\}, q, \pi)$ in $\mathcal{Y}$. $\square$

4.6. Thus the map $(\tilde{C}, \{\tilde{p}_m^\alpha\}, \{r^k\}, \pi)$ must lie in $\mathcal{V}^{d,p_a(\tilde{C})}(\vec{h'})$ where $\vec{h'}$ is the partition corresponding to $(\pi|_{\tilde{C}})^*(\infty)$. By (3), $\tilde{C}$ moves in a family of dimension at most

$$d + 2p_a(\tilde{C}) - 2 + \left(\sum \tilde{h}_m + s\right),$$

and the curve $C(0)$ (as a nodal curve with marked points $\{p_m^\alpha(0)\}_{m,\alpha\in\Gamma_m(0)}$, $\{r^k\}_{1\leq k\leq s}$, and q) moves in a family of dimension at most

$$3p_a(C(0)) - 3 + \sum h_m(0) + s + 1,$$

so $\mathcal{Y}$ is contained in a family of dimension

$$\begin{aligned} \left(d + 2p_a(\tilde{C}) - 2 + \sum \tilde{h}_m + s\right) &+ \left(3p_a(C(0)) - 3 + \sum h_m(0) + s + 1\right) \\ &= d + 2g - 1 + \sum h_m - 1 + p_a(C(0)) \\ &= \dim \mathcal{V} - 1 + p_a(C(0)) \qquad (4) \end{aligned}$$

by (3).

4.7. *Components* $\mathcal{Y}$ *of* D_∞ *satisfying* $p_a(C(0)) = 0$. For each choice of a partition $\Gamma_m = \Gamma_m(0) \coprod \tilde{\Gamma}_m$ (inducing a partition of h_m into $h_m(0) + \tilde{h}_m$), a positive integer s, and

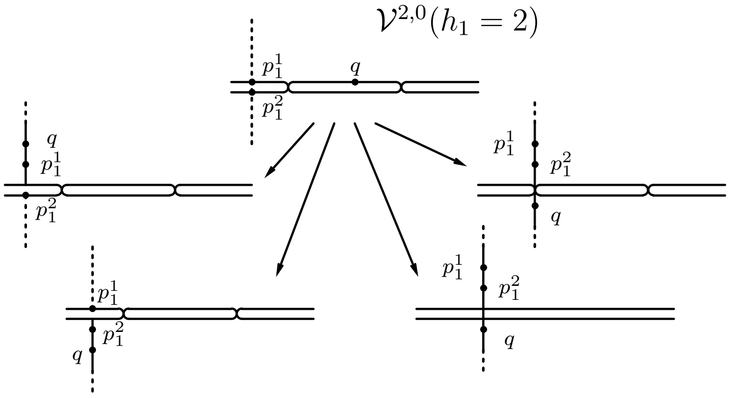

FIGURE 8. The four possible components of D_∞ on $\mathcal{V}^{2,0}(h_1 = 2)$

$(m^1, \ldots, m^s)$ satisfying $\sum m^k + \sum m\tilde{h}_m = d$, consider the closure in $\overline{\mathcal{M}}_{g,\sum h_m+1}(\mathbb{P}^1, d)^Q$ of points corresponding to maps $(C(0) \cup \tilde{C}, \{p_m^\alpha\}, q, \pi)$ where

- The curve $C(0) \cup \tilde{C}$ is nodal, and the curves $C(0)$ and $\tilde{C}$ intersect at the points $\{r^k\}$.
- The curve $C(0)$ is isomorphic to $\mathbb{P}^1$, has labelled points $\{p_m^\alpha(0)\}$ and q, and $\pi(C(0)) = \infty$.
- The curve $\tilde{C}$ is smooth of arithmetic genus $g - s + 1$ with labelled points $\{\tilde{p}_m^\alpha\}$. The map π is degree d on $\tilde{C}$, and

$$(\pi|_{\tilde{C}})^*(\infty) = \sum m\tilde{p}_m^\alpha + \sum m^k r^k.$$

Let $\mathcal{U}$ be the union of these substacks (over all choices of partitions of Γ_m); $\dim \mathcal{U} = \dim \mathcal{V} - 1$ by (4).

An irreducible component $\mathcal{Y}$ of the divisor D_∞ satisfying $p_a(C(0)) = 0$ has dimension $\dim \mathcal{V} - 1$ and is a closed substack of $\mathcal{U}$, which also has dimension $\dim \mathcal{V} - 1$. Hence $\mathcal{Y}$ must be a component of $\mathcal{U}$ and the stable map corresponding to a general point of $\mathcal{Y}$ satisfies the three properties listed in the previous paragraph. (We don't yet know that all such $\mathcal{Y}$ are subsets of $\mathcal{V}$, but this will follow from Proposition 4.8 below.)

For example, if $d = 2$, $g = 0$, and $h_1 = 2$, there are four components of $\mathcal{U}$ (see Figure 8; $\pi^{-1}(\infty)$ is indicated by a dashed line). The components (from left to right) are a subset of the following.

1. The curve $\tilde{C}$ is irreducible and maps with degree 2 to $\mathbb{P}^1$, ramifying over two general points of $\mathbb{P}^1$. The marked points q and p_1^1 lie on $C(0)$, and p_1^2 lies on $\tilde{C}$. The curve $C(0)$ is attached to $\tilde{C}$ at the point
$$(\pi|_{\tilde{C}})^{-1}(\infty) \setminus \{p_1^2\}.$$
2. This case is the same as the previous one with p_1^1 and p_1^2 switched.
3. The curve $\tilde{C}$ is the disjoint union of two $\mathbb{P}^1$'s, each mapping to $\mathbb{P}^1$ with degree 1. Both intersect $C(0)$, which contains all the marked points.
4. The curve $\tilde{C}$ is irreducible and maps with degree 2 to $\mathbb{P}^1$, and one of its branch points is ∞. All of the marked points lie on $C(0)$.

Given a component $\mathcal{Y}$ of $\mathcal{U}$, we can determine the multiplicity of the divisor D_∞ along $\mathcal{Y}$. As this multiplicity will turn out to be positive, $\mathcal{Y}$ is a subset of $\mathcal{V}$, so as sets, $\mathcal{U} \subset D_\infty$.

4.8. *Proposition.* — *Fix such a component $\mathcal{Y}$ with $p_a(C(0)) = 0$. The multiplicity of D_∞ along $\mathcal{Y}$ is $\prod_{k=1}^s m^k$.*

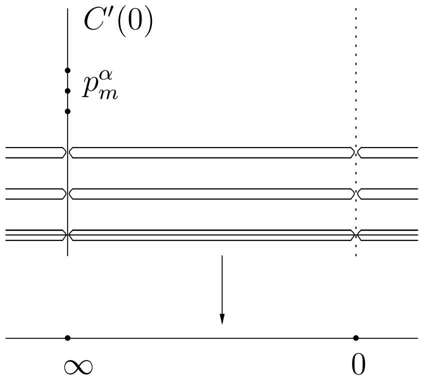

FIGURE 9. Step 2: It suffices to study deformations of a simpler map

For example, in Figure 8, the first 3 components appear with multiplicity 1, and the fourth with multiplicity 2.

Proof. We make a series of reductions to simplify the proof.

Step 1: Deformations of A_1. The multiplicity may be computed on a formal neighborhood of a general point of $\mathcal{Y}$. By Proposition 4.3, this space is naturally a product of the hulls of the special loci. If A_1 is the special locus containing q, then D_∞ is the pullback of a Cartier divisor on Def_{A_1}, and $\mathcal{Y}$ is the pullback of a Weil divisor on Def_{A_1}. Thus we need only consider deformations of the special locus A_1.

Step 2: Simpler maps. Fix a point $0 \in \mathbb{P}^1$ (distinct from ∞). Let $\mathcal{V}'$ be the closure in $\overline{\mathcal{M}}_{0,|\Gamma(0)|+1+s}(\mathbb{P}^1, \sum_{k=1}^s m^k)$ of points representing maps $(C, \{p_m^\alpha(0)\}, q, \{y^k\}_{k=1}^s, \pi)$ where $\pi^*(\infty) = \sum m p_m^\alpha(0)$ and $\pi^*(0) = \sum m^k y^k$. Let $\mathcal{Y}'$ be the closure of points representing maps from a nodal curve $C'(0) \cup C(1) \cup \cdots \cup C(s)$, where

- $C'(0)$ is glued to $C(k)$ at a point (call it r^k) $(1 \leq k \leq s)$
- The marked curve $(C'(0), \{p_m^\alpha(0)\}, q, \{r^k\})$ is isomorphic to the marked curve $(C(0), \{p_m^\alpha(0)\}, q, \{r^k\})$, and is collapsed to ∞ by π.
- $C(k)$ maps to $\mathbb{P}^1$ with degree m^k, and is totally ramified over 0 (at y^k) and ∞ (at r^k).

(See Figure 9.) (An étale neighborhood of) the special locus $C'(0)$ of a general map in $\mathcal{Y}'$ is isomorphic to (an étale neighborhood of) A_1. As the only other special loci of such a map are the points y^k, the formal deformations in $\mathcal{V}'$ of a general map in $\mathcal{Y}'$ are given by Def_{A_1}. As it suffices to consider the case when $\Gamma = \Gamma(0)$ (all the marked points mapping to ∞ are on $C(0)$), we now assume that this is the case.

Step 3: Fixing the marked curve. There is a morphism of stacks $\alpha : \mathcal{V}' \to \overline{\mathcal{M}}_{0,\sum h_m+1+s}$ that sends each map to the stable model of the underlying pointed nodal curve. Given any smooth marked curve $(C(0), \{p_m^\alpha\}, q, \{y^k\})$ in $\overline{\mathcal{M}}_{0,\sum h_m+1+s}$, the stable map $(C, \{p_m^\alpha\}, q, \{y^k\}, \pi)$ defined in Step 2 corresponds to a point in $\alpha^{-1}(C(0), \{p_m^\alpha\}, q, \{y^k\})$, so $\alpha|_{\mathcal{Y}'}$ is surjective. Let $\mathcal{F}_\alpha$ be a general fiber of α. By Sard's theorem, $\alpha|_{\mathcal{Y}'}$ is regular in a Zariski-open subset of $\mathcal{Y}'$, so $[\mathcal{Y}'] \cap [\mathcal{F}_\alpha] = [\mathcal{Y}' \cap \mathcal{F}_\alpha]$ in the Chow group of $[\mathcal{V}']$.

In order to determine the multiplicity of $D_\infty|_{\mathcal{V}'}$ along $\mathcal{Y}'$, it suffices to determine the multiplicity of the Cartier divisor $D_\infty|_{\mathcal{F}_\alpha}$ along $\mathcal{Y}' \cap \mathcal{F}_\alpha$ (in the Chow group of $\mathcal{F}_\alpha$). (*Proof:* As D_∞ is a Cartier divisor, $[D_\infty|_{\mathcal{F}_\alpha}] = D_\infty \cdot [\mathcal{F}_\alpha]$. Thus if $[D_\infty|_{\mathcal{V}'}] = m[\mathcal{Y}']$ in $A^1\mathcal{V}'$ then, intersecting with $[\mathcal{F}_\alpha]$, $[D_\infty|_{\mathcal{F}_\alpha}] = D_\infty \cdot [\mathcal{F}_\alpha] = m[\mathcal{Y}'] \cdot [\mathcal{F}_\alpha] = m[\mathcal{Y}' \cap \mathcal{F}_\alpha]$ in $A^1\mathcal{F}_\alpha$.)

With this in mind, fix a general $(C, \{p_m^\alpha\}, q, \{y^k\})$ in $\overline{\mathcal{M}}_{0,\sum h_m+1+s}$ and let $\mathcal{V}_o''$ be the points of $\overline{\mathcal{M}}_{0,\sum h_m+1+s}(\mathbb{P}^1, d)^Q$ representing stable maps $(C, \{p_m^\alpha\}, q, \{y^k\}, \pi)$ where $\pi^*(\infty) = \sum m p_m^\alpha$

and $\pi^*(0) = \sum m^k y^k$. Let f and g be sections of $\mathcal{O}_C(d)$ with associated divisors

$$(f) = \sum m p_m^\alpha, \quad (g) = \sum m^k y^k.$$

Then the maps in $\mathcal{V}_o''$ are those of the form $[\beta f, \gamma g]$ where

$$[\beta, \gamma] \in \mathbb{P}^1 \setminus \{[0,1],[1,0]\}$$

where $\infty = [0,1]$ and $0 = [1,0]$.

Let $\mathcal{V}'' = \mathcal{V}' \cap \mathcal{F}_\alpha$ be the closure of $\mathcal{V}_o''$, and define $\mathcal{Y}'' = \mathcal{Y} \cap \mathcal{F}_\alpha$ similarly. Let V'' be the course moduli scheme of $\mathcal{V}''$, and Y'' the course moduli scheme of $\mathcal{Y}''$.

Step 4: V'' is isomorphic to $\mathbb{P}^1$. The variety V'' is proper, and $\mathcal{V}_o'' \cong \mathbb{P}^1 \setminus \{[0,1],[1,0]\}$, so the normalization of the variety V'' is $\mathbb{P}^1$.

The evaluation map gives a morphism from the curve V'' to $\mathbb{P}^1$:

$$[(C, \{p_m^\alpha\}, q, \{y^k\}, \pi)] \to \pi(q)$$

and this map is an isomorphism from $\mathcal{V}_o''$ to $\mathbb{P}^1 \setminus \{[0,1],[1,0]\}$, so it must be an isomorphism from V'' to $\mathbb{P}^1$.

Step 5: Calculating the multiplicity. Let w be a general point of the target $\mathbb{P}^1$. Then the divisor $\{\pi(q) = w\}$ is linearly equivalent to $D_\infty|_{V''} = ev_q^* \infty|_{V''} = \{\pi(q) = \infty\}|_{V''}$, and is $\mathcal{O}_{V''}(1)$.

Thus, on V'', $D_\infty|_{V''} = [1,0] = Y''$. But the limit map has automorphism group

$$\mathbb{Z}/m^1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/m^s\mathbb{Z}$$

(as $\operatorname{Aut}(C(k), \pi|_{C(k)}) = m^k$) so *as stacks* $[D_\infty|_{\mathcal{V}''}] = \left(\prod m^k\right)[\mathcal{Y}'']$. Therefore $[D_\infty] = \prod m^k [\mathcal{Y}]$. □

In order to extend these results to components for which $p_a(C(0)) = 1$, we will need the following result.

4.9. *Proposition. — Let $\mathcal{Y}$ be a component of D_∞, with $(C, \{p_m^\alpha\}, q, \pi)$ the map corresponding to a general point of $\mathcal{Y}$, $C(0) \cap \tilde{C} = \{r^1, ..., r^s\}$, and m^k the multiplicity of $\pi^*(\infty)$ on $\tilde{C}$ at r^k. Then*

$$\mathcal{O}_{C(0)}\left(\sum_{m,\alpha} m p_m^\alpha(0)\right) \cong \mathcal{O}_{C(0)}\left(\sum_{k=1}^s m^k r^k\right)$$

where $\Gamma(0) \subset \Gamma$ are the marked points whose limits lie in $C(0)$.

Proof. For a map $(C, \{p_m^\alpha\}, q, \pi)$ corresponding to a general point in $\mathcal{V}$, we have the following relation in the Picard group of C:

$$\pi^*(\mathcal{O}_{\mathbb{P}^1}(1)) \cong \mathcal{O}_C(\sum_{m,\alpha} m p_m^\alpha).$$

Thus for the curve corresponding to a general point of $\mathcal{Y}$ the invertible sheaf $\mathcal{O}_C(\sum_{m,\alpha} m p_m^\alpha)$ must be a possible limit of $\pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$. The statement of the lemma depends only on an étale neighborhood of $C(0)$, so we may assume (as in Step 2 of the proof of Proposition 4.8) that $\Gamma = \Gamma(0)$, and $\tilde{C}$ consists of k rational tails $C(1)$, ... , $C(k)$ each totally ramified where they intersect $C(0)$. As the dual graph of C is a tree, C is of compact type (i.e. $\operatorname{Pic}^0 C$ is compact). One possible limit of $\pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ is the line bundle that is trivial on $C(0)$ and degree m^k on $C(k)$. If a curve C' is the central fiber of a one-dimensional family of curves, and $C' = C_1 \cup C_2$, and a line bundle $\mathcal{L}$ is the limit of a family of line bundles, then the line bundle $\mathcal{L}'$ whose restriction to C_i is $\mathcal{L}|_{C_i}((-1)^i C_1 \cap C_2)$ is another possible limit. Thus the line bundle that is trivial on $\tilde{C}$ and $\mathcal{O}_C(\sum m^k r^k)$ on $C(0)$ is a possible limit of $\pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$.

If two line bundles on a curve C of compact type are possible limits of the same family of line bundles, and they agree on all components but one of C, then they must agree on the remaining component. But $\mathcal{O}_C(\sum mp^\alpha_m)$ is another limit of $\pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$ that is trivial on $\tilde{C}$, so the result follows. $\square$

4.10. We now determine all components $\mathcal{Y}$ of D_∞ satisfying $p_a(C(0)) = 1$. For each choice of a partition of Γ_m into $\Gamma_m(0) \coprod \tilde{\Gamma}_m$, a positive integer s, and $(m^1, \dots, m^s)$ satisfying $\sum m^k + \sum m\tilde{h}_m = d$, consider the closure in $\overline{\mathcal{M}}_{g,\sum h_m+1}(\mathbb{P}^1, d)$ of points corresponding to maps

$$(C(0) \cup \tilde{C}, \{p^\alpha_m\}, q, \pi)$$

where

1. The curve $C(0) \cup \tilde{C}$ is nodal, and $C(0)$ and $\tilde{C}$ intersect at the points $\{r^k\}$.
2. The curve $C(0)$ is a smooth genus 1 curve with labelled points $\{p^\alpha_m(0)\}$ and q, where $\mathcal{O}_{C(0)}(\sum mp^j_m(0)) \cong \mathcal{O}_{C(0)}(\sum m^k r^k)$, and $\pi(C(0)) = \infty$.
3. The curve $\tilde{C}$ is smooth of arithmetic genus $g - s$ with labelled points $\{\tilde{p}^\alpha_m\}$. The map π is degree d on $\tilde{C}$, and

$$(\pi|_{\tilde{C}})^*(\infty) = \sum m\tilde{p}^\alpha_m + \sum m^k r^k.$$

Let $\mathcal{U}$ be the union of all such substacks (over all choices of s, partitions of Γ, etc.). The divisorial condition $\mathcal{O}_{C(0)}(\sum mp^\alpha_m(0)) \cong \mathcal{O}_{C(0)}(\sum m^k r^k)$ defines a substack $\mathcal{M}'$ of pure codimension 1 in $\mathcal{M}_{1,\sum h_m+1+s}$: for any

$$(C, \{p^\alpha_m\}, q, \{r^k\}_{k>1}) \in \mathcal{M}_{1,\sum h_m+1+(s-1)}$$

the subscheme of points $r^1 \in C$ satisfying

$$\mathcal{O}_C(m^1 r^1) \cong \mathcal{O}_C\left(\sum mp^\alpha_m(0) - \sum_{k>1} m^k r^k\right)$$

is reduced of degree $(m^1)^2$. Thus the stack $\mathcal{M}'$ is a degree $(m^1)^2$ étale cover of $\mathcal{M}_{1,\sum h_m+1+(s-1)}$. By this observation and (4), $\mathcal{U}$ has pure dimension $\dim \mathcal{V} - 1$.

An irreducible component $\mathcal{Y}$ of the divisor D_∞ satisfying $p_a(C(0)) = 1$ has dimension $\dim \mathcal{V} - 1$ and is a substack of $\mathcal{U}$, which also has dimension $\dim \mathcal{V} - 1$. Hence $\mathcal{Y}$ must be a component of $\mathcal{U}$ and the stable map corresponding to a general point of $\mathcal{Y}$ satisfies properties 1–3 above.

The determination of multiplicity and local structure is identical to the genus 0 case.

4.11. *Proposition. — Fix such a component $\mathcal{Y}$ with $p_a(C(0)) = 1$. If $m^1, \dots, m^s$ are the multiplicities of $\pi^*(\infty)$ along $\tilde{C}$ at the s points $C(0) \cap \tilde{C}$, then this divisor appears with multiplicity $\prod_k m^k$.*

Proof. The proof is identical to that of Proposition 4.8. We summarize the steps here.

Step 1. If A_1 is the special locus of π containing q, then it suffices to analyze Def_{A_1}.

Step 2. We may consider instead deformations of the map consisting of $C(0)$, with s rational tails ramifying completely over ∞ (at points r^k) and over another point 0 (at points y^k). In particular, we assume $\Gamma = \Gamma(0)$.

Step 3. Let $\overline{\mathcal{M}}'_{1,\sum h_m+1+s}$ be the substack of $\overline{\mathcal{M}}_{1,\sum h_m+1+s}$ that is the closure of the set of points representing smooth stable curves where $\mathcal{O}(\sum mp^\alpha_m) \cong \mathcal{O}(\sum m^k y^k)$. If α is defined by

$$\alpha : \mathcal{V}' \to \overline{\mathcal{M}}'_{1,\sum h_m+1+s},$$

then $\alpha|_{\mathcal{Y}}$ is dominant, so we may consider a fixed general stable curve

$$(C, \{p_m^\alpha\}, q, \{y^k\}) \in \overline{\mathcal{M}}'_{1,\sum h_m+1+s}.$$

Steps 4 and 5. The variety V'' is $\mathbb{P}^1$, and the multiplicity calculation is identical. □

4.12. We have now found all components of D_∞ on $\mathcal{X}_1(d,\Gamma,\Delta)$ and $\mathcal{W}_1(d,\Gamma,\Delta)$ where $\Delta = \{\mathbb{P}^1\}$ (as in these cases $p_a(C(0)) \leq 1$), and the multiplicity of D_∞ along each component. We summarize this in two theorems which will be invoked later.

4.13. *Theorem (Genus 0 maps to $\mathbb{P}^1$). — The components of D_∞ on $\mathcal{X}_1(d,\Gamma,\Delta)$ are of the form*

$$\mathcal{Y}_1(0,\Gamma(0),\Delta; d(1),\Gamma(1),\emptyset; \ldots; d(l),\Gamma(l),\emptyset)$$

for some positive integer l and partitions $d = \sum_{k=1}^l d(k)$, $\Gamma = \coprod_{k=0}^l \Gamma(k)$. If $m^k = d(k) - \sum_m m|\Gamma_m(k)|$ (as in Section 3.6), this component appears with multiplicity $\prod_k m^k$.

4.14. *Theorem (Genus 1 maps to $\mathbb{P}^1$). — Let $\mathcal{Y}$ be a component of D_∞ on $\mathcal{W}_1(d,\Gamma,\Delta)$. Fix a positive integer l and partitions $d = \sum_{k=1}^l d(k)$ and $\Gamma_m = \coprod_{k=0}^l \Gamma_m(k)$. Then $\mathcal{Y}$ is a component of*

$$\mathcal{Y}_1^a(0,\Gamma(0),\Delta; d(1),\Gamma(1),\emptyset; \ldots; d(l),\Gamma(l),\emptyset),$$
$$\mathcal{Y}_1^b(0,\Gamma(0),\Delta; d(1),\Gamma(1),\emptyset; \ldots; d(l),\Gamma(l),\emptyset)_{m_1^1} \text{ (for some } m_1^1\text{), or}$$
$$\mathcal{Y}_1^c(0,\Gamma(0),\Delta; d(1),\Gamma(1),\emptyset; \ldots; d(l),\Gamma(l),\emptyset).$$

If $m^k = d(k) - \sum_m m|\Gamma_m(k)|$, the components of the first and third types appear with multiplicity $\prod_{k=1}^l m^k$ and those of the second type appear with multiplicity $m_1^1(m^1 - m_1^1)\prod_{k=2}^l m^k$.

In all cases, the multiplicity is the product of the "new ramifications" of the components not mapped to ∞.

For general g, the above argument identifies some of the components of D_∞, but further work is required to determine what happens when $p_a(C(0)) > 1$.

4.15. Aside: local structure near D_∞, and pathological behavior of $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$. The proofs of Propositions 4.8 and 4.11 can be refined to determine the local structure of $\mathcal{V}$ near $\mathcal{Y}$ in both cases. As these results will not be needed, the proof is omitted.

4.16. *Corollary. — Let $\mathcal{Y}$ be the same component as in Propositions 4.8 or 4.11. A formal neighborhood of a general point of $\mathcal{Y}$ in the stack $\mathcal{V}$ is isomorphic to*

$$\operatorname{Spf} \mathbb{C}[[a, b_1, \ldots, b_s, c_1, \ldots, c_{\dim \mathcal{V}-1}]]/(a = b_1^{m^1} = \cdots = b_s^{m^s})$$

with D_∞ given by $(a = 0)$, and $\mathcal{Y}$ given set-theoretically by the same equation.

In particular, if $\gcd(m^i, m^j) > 1$ for some i and j, $\mathcal{V}$ fails to be unibranch at a general point of $\mathcal{Y}$. Similar phenomena occur in other situations ([CH] Proposition 4.8, [V3] Section 2.5, etc.), although the proofs seem unrelated.

4.17. Let $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)^o$ be the closure of points corresponding to maps from irreducible genus g curves. One might hope that $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)^o$ is smooth for general g and d. This is not the case. The phenomenon of Proposition 4.16 suggested the following example of a map to $\mathbb{P}^1$ that can be smoothed in 2 different ways.

The dimension of $\overline{\mathcal{M}}_4(\mathbb{P}^1, 4)^o$ is 14. Consider the family $\mathcal{Y}$ of stable maps whose general element parametrizes a smooth genus 3 curve $C(0)$ meeting a rational tail $C(1)$ at two general points. The curve $C(0)$ maps with degree 0 to $\mathbb{P}^1$, and the rational tail maps with degree 4 to $\mathbb{P}^1$, ramifying at both points of intersection with $C(0)$.

The substack $\mathcal{Y}$ has dimension 13: 8 for the two-pointed genus 3 curve $C(0)$, 1 for the image of $C(0)$ in $\mathbb{P}^1$, and 4 for the other ramification points of $C(1)$. Thus if $\mathcal{Y}$ is contained in $\overline{\mathcal{M}}_4(\mathbb{P}^1, 4)^o$, it is a Weil divisor.

4.18. *Proposition.* — *$\overline{\mathcal{M}}_4(\mathbb{P}^1, 4)^o$ has two smooth branches along $\mathcal{Y}$, intersecting transversely.*

By a similar argument, we can find a codimension 1 unibranch singularity of $\overline{\mathcal{M}}_5(\mathbb{P}^1, 5)^o$, and singularities of $\overline{\mathcal{M}}_8(\mathbb{P}^1, 7)^o$ with several codimension 1 singular branches.

4.19. Maps of genus 1 curves to $\mathbb{P}^1$. We conclude with necessary results unrelated to the earlier part of this section.

4.20. *Lemma.* — *Suppose $\pi : C \to \mathbb{P}^1$ is a map from a connected nodal curve of arithmetic genus 1.*

(a) *If π contracts no component of arithmetic genus 1, then*

$$h^1(C, \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))) = h^1(C, \pi^*(\mathcal{O}_{\mathbb{P}^1}(2))) = 0.$$

(b) *If C has a contracted component E of arithmetic genus 1, where E intersects the rest of the components R at two points p and q (and possibly others) with $\pi|_R$ étale at p, then $h^1(C, \pi^*(\mathcal{O}_{\mathbb{P}^1}(2))) = 1$.*

By "contracting no component of C of arithmetic genus 1" we mean that all connected unions of contracted irreducible components of C have arithmetic genus 0.

Proof. (a) By Serre duality, it suffices to show that

$$H^0(C, \mathcal{K}_C \otimes \pi^*(\mathcal{O}(-1))) = 0.$$

Assume otherwise that such (C, π) exists, and choose one with the fewest components, and choose a nonzero global section s of $\mathcal{K}_C \otimes \pi^*(\mathcal{O}(-1))$. If $C = C' \cup R$ where R is a rational tail (intersecting C' at one point), then $s = 0$ on R as

$$\deg_R(\mathcal{K}_C \otimes \pi^*(\mathcal{O}(-1))) = -1 - \deg_\pi R < 0.$$

Then $s|_{C'}$ is a section of $(\mathcal{K}_C \otimes \pi^*(\mathcal{O}(-1)))|_{C'}$ that vanishes on $C' \cap R$. But $\mathcal{K}_{C'} = \mathcal{K}_C(-C' \cap R)|_{C'}$, so this induces a non-zero section of $\mathcal{K}_{C'} \otimes (\pi|_{C'})^*(\mathcal{O}(-1))$, contradicting the minimality of the number of components. Thus C has no rational tails, and C is either an irreducible genus 1 curve or a cycle of rational curves. If C is an irreducible genus 1 curve, then C isn't contracted by hypothesis, so $\mathcal{K}_C \otimes \pi^*(\mathcal{O}(-1))$ is negative on C as desired. If C is a cycle $C_1 \cup \cdots \cup C_s$ of $\mathbb{P}^1$'s, then

$$\deg_{C_i}(\mathcal{K}_C \otimes \pi^*(\mathcal{O}(-1))) = -\deg C_i \leq 0.$$

As one of the curves has positive degree, there are no global sections of $\mathcal{K}_C \otimes \pi^*(\mathcal{O}(-1))$.

(b) The proof is essentially the same, and is omitted. □

4.21. *Proposition.* — *Suppose $(C, \{p_i\}, \pi)$ is a stable map in $\overline{\mathcal{M}}_{1,m}(\mathbb{P}^1, d)$ satisfying*

(a) *C has no contracted component of arithmetic genus 1, or*

(b) *C has a contracted component E of arithmetic genus 1, where E intersects the rest of the components R at two points p and q (and possibly others) with $\pi|_R$ étale at p.*

Then $\overline{\mathcal{M}}_1(\mathbb{P}^1, d)$ is smooth of dimension $2d + m$ at $(C, \{p_i\}, \pi)$.

The result is almost certainly true even without the étale condition in (b).

Proof. (a) As $H^1(C, \pi^* T_{\mathbb{P}^1}) = 0$ by the previous lemma, $\overline{\mathcal{M}}_{1,m}(\mathbb{P}^1, d)$ is smooth of dimension $\deg \pi^* T_{\mathbb{P}^1} + m = 2d + m$. The argument is well-known, but for completeness we give it here.

From the exact sequence for infinitesimal deformations of stable maps (see Appendix A), we have

$$
\begin{array}{rcl}
0 & \longrightarrow & \operatorname{Aut}(C, \{p_i\}) \longrightarrow H^0(C, \pi^* T_{\mathbb{P}^1}) \\
\longrightarrow \operatorname{Def}(C, \{p_i\}, \pi) & \longrightarrow & \operatorname{Def}(C, \{p_i\}) \longrightarrow H^1(C, \pi^* T_{\mathbb{P}^1}) \\
\longrightarrow \operatorname{Ob}(C, \{p_i\}, \pi) & \longrightarrow & 0
\end{array} \tag{5}
$$

where $\operatorname{Aut}(C, \{p_i\})$ (resp. $\operatorname{Def}(C, \{p_i\})$) are the infinitesimal automorphisms (resp. infinitesimal deformations) of the marked curve, and $\operatorname{Def}(C, \{p_i\}, \pi)$ (resp. $\operatorname{Ob}(C, \{p_i\}, \pi)$) are the infinitesimal deformations (resp. obstructions) of the stable map. As $H^1(C, \pi^* T_{\mathbb{P}^1}) = 0$, $\operatorname{Ob}(C, \{p_i\}, \pi) = 0$ from (5). Thus the deformations of $(C, \{p_i\}, \pi)$ are unobstructed, and the dimension follows from:

$$
\begin{aligned}
\dim \operatorname{Def}(C, \{p_i\}, \pi) - \dim \operatorname{Ob}(C, \{p_i\}, \pi) &= (\dim \operatorname{Def}(C, \{p_i\}) - \dim \operatorname{Aut}(C, \{p_i\})) \\
&\quad + (h^0(C, \pi^* T_{\mathbb{P}^1}) - h^1(C, \pi^* T_{\mathbb{P}^1})) \\
&= m + 2d.
\end{aligned}
$$

(b) For convenience (and without loss of generality) assume $m = 0$. As $h^1(C, \pi^* T_{\mathbb{P}^1}) = 1$ (previous Lemma), our proof of (a) will not carry through. However, $\operatorname{Def}(C, \{p_i\}, \pi)$ does not surject onto $\operatorname{Def}(C, \{p_i\})$ in long exact sequence (5), as it is not possible to smooth the nodes independently: one cannot smooth the node at p while preserving the other nodes even to first order. (This is well-known; one argument, due to M. Thaddeus, is to consider a stable map (C, π) in $\overline{\mathcal{M}}_1(\mathbb{P}^1, 1)$ and express the obstruction space $\mathbb{E}\mathrm{xt}^2(\underline{\Omega}_\pi, \mathcal{O}_C)$ as the dual of $H^0(C, \mathcal{F})$ for a certain sheaf $\mathcal{F}$, [Th].) Thus the map $\operatorname{Def}(C) \to H^1(C, \pi^* T_{\mathbb{P}^1})$ is not the zero map, so $\operatorname{Def}(C)$ surjects onto $H^1(C, \pi^* T_{\mathbb{P}^1})$. Therefore $\operatorname{Ob}(C, \pi) = 0$, so the deformations are unobstructed.

The rest of the proof is identical to that of (a) □

5. Maps to $\mathbb{P}^n$

We begin with generalities about curves in projective space justifying Reductions A–C in Section 3.3.

The following proposition is a straightforward consequence of the Kleiman-Bertini theorem for stacks. (Kleiman's original proof [K1] carries through completely in the category of Deligne-Mumford stacks.)

5.1. *The Kleiman-Bertini Theorem. — Let $\mathcal{A}$ be a reduced irreducible substack of $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^n, d)$, and let p be one of the labelled points. Then there is a Zariski-open subset U of the dual projective space $(\mathbb{P}^n)^*$ such that for all $[H'] \in U$ the intersection $\mathcal{A} \cap ev_p^* H'$, if nonempty, is reduced of dimension* $\dim \mathcal{A} - 1$.

Let $\mathcal{B}$ be a proper closed substack of $\mathcal{A}$. Then there is a Zariski-open subset U' of the dual projective space $(\mathbb{P}^n)^*$ such that for all $[H'] \in U'$, each component of $\mathcal{B} \cap ev_p^* H'$ is a proper closed substack of a component of $\mathcal{A} \cap ev_p^* H'$.

5.2. *Proposition. — Let $\mathcal{A}$ be an irreducible family of stable maps where the source curve is of constant topological type and the components are distinguished. Suppose E is a contracted genus g component with s special points (i.e. marked points or branches of nodes). If the induced map $i : \mathcal{A} \to \overline{\mathcal{M}}_{g,s}$ is non-constant then $\mathcal{A}$ is stably enumeratively irrelevant (3.5).*

Proof. Each component of the universal curve over $\mathcal{A}$ also satisfies the hypotheses of the Proposition. Hence by replacing $\mathcal{A}$ with $\mathcal{A}^{(j)}$ for suitable j, it suffices to prove that $\mathcal{A}$ is enumeratively irrelevant. Suppose otherwise that $\mathcal{A}$ were enumeratively relevant.

If $D = ev_{p_j}^* H'$ (p_j one of the marked points, H' a general hyperplane), then the hypothesis also holds for each component of D. (Reason: if p is a point of $\overline{\mathcal{M}}_{g,s}$, then each component of i^*p is a proper substack. By Kleiman-Bertini 5.1, D contains no component of i^*p, so $i^*p \cap D$ is of codimension at least 2 in $\mathcal{A}$. Thus $i(D)$ is non-constant.)

As $\mathcal{A}$ is enumeratively relevant, some component of $ev_j^* H$ (for some j) is enumeratively relevant (and can be represented by a family also satisfying the hypotheses of the Proposition). By repeating this process $\dim \mathcal{A}$ times, we are left with an irreducible dimension 0 stack also satisfying the hypotheses of the Proposition. This is impossible, as a map from a point to $\overline{\mathcal{M}}_{g,m}$ must be constant. □

We next justify the Reduction steps described in Section 3.3, A and B in the next Proposition, and C in Proposition 5.5.

5.3. *Proposition. — Fix d, Γ, and Δ. Let H' be a general hyperplane in $\mathbb{P}^n$. Then:*

(a) $\mathcal{X}(d,\Gamma,\Delta \cup \{\mathbb{P}^n\})$ *(resp. $\mathcal{W}(d,\Gamma,\Delta \cup \{\mathbb{P}^n\})$) is the universal curve over $\mathcal{X}(d,\Gamma,\Delta)$ (resp. $\mathcal{W}(d,\Gamma,\Delta)$).*

(b) *Fix m and $\alpha \in \Gamma_m$. Suppose Γ' is the same as Γ except ${\Gamma'}_m^\alpha = \Gamma_m^\alpha \cap H'$. Then the Cartier divisor $ev_{p_m^\alpha}^* H'$ on $\mathcal{X}(d,\Gamma,\Delta)$ (resp. $\mathcal{W}(d,\Gamma,\Delta)$) is $\mathcal{X}(d,\Gamma',\Delta)$ (resp. $\mathcal{W}(d,\Gamma',\Delta)$).*

(c) *Fix $\alpha \in \Delta$. Suppose Δ' is the same as Δ except ${\Delta'}^\alpha = \Delta^\alpha \cap H'$. Then the Cartier divisor $ev_{q^\alpha}^* H'$ on $\mathcal{X}(d,\Gamma,\Delta)$ (resp. $\mathcal{W}(d,\Gamma,\Delta)$, $\mathcal{Z}(d,\Delta)$) is $\mathcal{X}(d,\Gamma,\Delta')$ (resp. $\mathcal{W}(d,\Gamma,\Delta')$, $\mathcal{Z}(d,\Delta')$).*

Proof. (a) follows from

$$\mathcal{X}(d,\Gamma,\Delta \cup \{\mathbb{P}^n\}) = \mathcal{X}(d,\Gamma,\Delta) \times_{\overline{\mathcal{M}}_{0,\sum|\Gamma_m|+|\Delta|}(\mathbb{P}^n,d)} \overline{\mathcal{M}}_{0,\sum|\Gamma_m|+|\Delta|+1}(\mathbb{P}^n,d)$$

(and the analogous statement for $\mathcal{W}$).

(b) Clearly

$$\mathcal{X}(d,\Gamma',\Delta) \subset ev_{p_m^\alpha}^* H';$$

each component of $\mathcal{X}(d,\Gamma',\Delta)$ appears with multiplicity one by Kleiman-Bertini 5.1. The only other possible components of $ev_{p_m^\alpha}^* H'$ are those whose general point represents a map where $\pi^{-1}H$ is not a union of points (i.e. contains a component of C). But such maps form a union of proper subvarieties of components of $\mathcal{X}(d,\Gamma,\Delta)$, and by Kleiman-Bertini 5.1 such maps cannot form a component of $ev_{p_m^\alpha}^* H' \cap \mathcal{X}(d,\Gamma,\Delta)$.

Replacing p_m^α with q^α in the previous paragraph gives a proof of (c) for $\mathcal{X}$. The same arguments hold with $\mathcal{X}$ replaced by $\mathcal{W}$, and for (c), $\mathcal{Z}$. □

5.4. Next, we justify Reduction C.

Let A be a general $(n-2)$-plane in H. The projection p_A from A induces a rational map $\rho_A : \overline{\mathcal{M}}_{g,m}(\mathbb{P}^n,d) \dashrightarrow \overline{\mathcal{M}}_{g,m}(\mathbb{P}^1,d)$, that is a morphism (of stacks) at points representing maps $(C,\{p_i\},\pi)$ whose image $\pi(C)$ does not meet A. Let $\mathcal{V} \subset \overline{\mathcal{M}}_{g,m}(\mathbb{P}^n,d)$ be the open substack corresponding to maps where $\pi^{-1}A$ is a union of reduced points distinct from the m marked points $\{p_i\}$.

5.5. *Proposition. —*

(a) *The morphism ρ_A can be extended to $\mathcal{V}$, where the image of a map $(C, \{p_i\}, \pi) \in \mathcal{V}$ is the stable map $(\tilde{C}, \{p_i\}, \pi_{\mathbb{P}^1})$, with*

$$\tilde{C} = C \cup C_1 \cup \cdots \cup C_{\#\pi^{-1}A},$$

where $C_1, \ldots, C_{\#\pi^{-1}A}$ are rational tails attached to C at the points of $\pi^{-1}A$,

$$\pi_{\mathbb{P}^1} \mid_{\{C \setminus \pi^{-1}A\}} = (p_A \circ \pi) \mid_{\{C \setminus \pi^{-1}A\}}$$

(which extends to a morphism from all of C) and $\pi_{\mathbb{P}^1} \mid_{C_k}$ is a degree 1 map to $\mathbb{P}^1$ ($1 \leq k \leq \#\pi^{-1}A$).

(b) *If $g = 0$, then ρ_A is a smooth morphism of stacks (on $\mathcal{V}$) of relative dimension $(n-1)(d+1)$.*

(c) *If $g = 1$ and π doesn't collapse any component of arithmetic genus 1, then ρ_A is a smooth morphism of stacks (at $(C, \{p_i\}, \pi)$) of relative dimension $(n-1)d$.*

5.6. To show that a morphism of stacks $\mathcal{A} \to \mathcal{B}$ is smooth at a point $a \in \mathcal{A}$, where $\mathcal{B}$ is smooth and $\mathcal{A}$ is equidimensional, it suffices to show that the fiber is smooth at a, or equivalently that the Zariski tangent space to the fiber at a is of dimension $\dim \mathcal{A} - \dim \mathcal{B}$.

Proof. Let $\nu : Bl \to \mathbb{P}^n$ be the blow-up of $\mathbb{P}^n$ along A, with exceptional divisor E. If $[L] \in H_2(\mathbb{P}^n)$ is the class of a line, let $\beta = d\nu^*[L] \in H_2(Bl)$. Let $\mathcal{V}'$ be the open substack of $\overline{\mathcal{M}}_{g,m}(Bl, \beta)$ corresponding to maps $(\tilde{C}, \{p_i\}, \pi_{Bl})$, where $s := \#\pi^{-1}A$,

$$\tilde{C} = C \cup C_1 \cup \cdots \cup C_s;$$

$C_i \cong \mathbb{P}^1$ ($1 \leq i \leq s$) and is mapped to E, isomorphically to fibers of ν; $(\pi_{Bl}|_C)^*E$ is a union of s reduced points, disjoint from $\{p_i\}$; and the C_i are glued to C at those s points.

Let $\mathcal{U}$ be the universal curve over $\overline{\mathcal{M}}_{g,m}(\mathbb{P}^n, d)$ (with natural map $\mathcal{U} \to \mathbb{P}^n$). Define $\mathcal{U}' := \mathcal{U} \times_{\mathbb{P}^n} Bl$; as A is a complete intersection, $\mathcal{U}'$ is the blow-up of $\mathcal{U}$ along the pullback of A. If $p \in \mathcal{V} \subset \overline{\mathcal{M}}_{g,m}(\mathbb{P}^n, d)$ is a closed point, $\mathcal{U}'_p$ is a curve $\tilde{C}$ as described in the previous paragraph. If $\mathcal{U}'|_{\mathcal{V}}$ is defined as $\mathcal{U}' \times_{\overline{\mathcal{M}}_{g,m}(\mathbb{P}^n,d)} \mathcal{V}$, then

$$\begin{array}{ccc} \mathcal{U}'|_{\mathcal{V}} & \to & Bl \\ \downarrow & & \\ \mathcal{V} & & \end{array}$$

is a family of stable maps to Bl. (*Proof:* All that needs to be verified is that $\mathcal{U}'|_{\mathcal{V}} \to \mathcal{V}$ is flat, which can be checked at closed points. Consider a point of $\pi^{-1}A$ on $\mathcal{U}$ mapping to $\mathcal{V}$, where the restriction of $\pi^{-1}A$ to the fiber is a closed point. Let $(S, \mathfrak{n})$ be the local ring of this point, and $(R, \mathfrak{m})$ be the local ring of the image in $\mathcal{V}$. Then $\pi^{-1}A$ defines an ideal of S generated by 2 elements (say x, y, so $(\mathfrak{m}S, x, y) = \mathfrak{n}$), and we can choose x and y so that neither x nor y vanishes to order 2 on the fiber (i.e. $\mathfrak{n} = (\mathfrak{m}S, x) = (\mathfrak{m}S, y)$). Then the flatness of the blow-up of (x, y) can be checked by looking at patches. Where $y \neq 0$, the patch is $\psi : \operatorname{Spec}(S[t]/(y - tx)) \to \operatorname{Spec} R$. As y isn't a 0-divisor on $S/\mathfrak{m}S$, $y - tx$ isn't a 0-divisor on $S[t]/\mathfrak{m}S[t]$. Hence by the local flatness criterion [Ma] Cor. to 22.5 p. 177, ψ is flat. The same argument holds with x and y interchanged.)

By the universal property of the moduli stack of stable maps, this induces a morphism $\mathcal{V} \to \overline{\mathcal{M}}_{g,m}(Bl, \beta)$; the image lies in $\mathcal{V}'$. On the other hand, the morphism $\nu : Bl \to \mathbb{P}^n$ induces a morphism

$$\overline{\mathcal{M}}_{g,m}(Bl, \beta) \to \overline{\mathcal{M}}_{g,m}(\mathbb{P}^n, d),$$

and the image of $\mathcal{V}'$ lies in $\mathcal{V}$. These morphisms clearly commute on the level of closed points, so $\mathcal{V} \cong \mathcal{V}'$.

Finally, the morphism $m : Bl \to \mathbb{P}^1$ (corresponding to projection from A) induces a morphism $\sigma : \mathcal{V}' \to \overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, d)$, proving (a). Let $(\tilde{C}, \{p_i\}, \pi_{\mathbb{P}^1})$ be the image of $(C, \{p_i\}, \pi)$ as in the statement of (a).

In short, we have diagrams as follows:

$$\begin{array}{ccc} \mathcal{V} & \xrightarrow{\sim} & \mathcal{V}' \\ & \rho_A \searrow & \downarrow \sigma \\ & & \overline{\mathcal{M}}_{g,m}(\mathbb{P}^1, d) \end{array} \qquad \begin{array}{ccc} (C, \{p_i\}, \pi) & \to & (\tilde{C}, \{p_i\}, \pi_{Bl}) \\ & \rho_A \searrow & \downarrow \sigma \\ & & (\tilde{C}, \{p_i\}, \pi_{\mathbb{P}^1}). \end{array}$$

In order to use Criterion 5.6, we compute the Zariski tangent space to the fiber of σ at a point of $\mathcal{V}'$. From the induced morphism of long exact sequences for infinitesimal deformations of maps (see Appendix A), we have

$$\begin{array}{ccccccc} \mathrm{Aut}(\tilde{C}, \{p_i\}) & \to & H^0(C, \pi_{Bl}^* T_{Bl}) & \to & \mathrm{Def}(\tilde{C}, \{p_i\}, \pi_{Bl}) & \to & \mathrm{Def}(\tilde{C}, \{p_i\}) \\ || & & \downarrow & & \downarrow & & || \\ \mathrm{Aut}(\tilde{C}, \{p_i\}) & \to & H^0(C, \pi_{\mathbb{P}^1}^* T_{\mathbb{P}^1}) & \to & \mathrm{Def}(\tilde{C}, \{p_i\}, \pi_{\mathbb{P}^1}) & \to & \mathrm{Def}(\tilde{C}, \{p_i\}) \end{array}$$

Hence the Zariski tangent space to the fiber $\ker(\mathrm{Def}(\tilde{C}, \{p_i\}, \pi_{Bl}) \to \mathrm{Def}(\tilde{C}, \{p_i\}, \pi_{\mathbb{P}^1}))$ is isomorphic to $\ker(H^0(C, \pi_{Bl}^* T_{Bl}) \to H^0(C, \pi_{\mathbb{P}^1}^* T_{\mathbb{P}^1}))$. These are first-order deformations of the map π_{Bl} preserving the image map $\pi_{\mathbb{P}^1}$ (keeping the source curve $\tilde{C}$ fixed).

As the image of the map $\pi_{Bl}|_{C_i}$ is a fiber of ν, it may move in E, but may not move out of E (even to first order, as the degree of $\pi_{Bl}^* E$ on C_i is -1). Thus deformations of π_{Bl} in $\ker(H^0(C, \pi_{Bl}^* T_{Bl}) \to H^0(C, \pi_{\mathbb{P}^1}^* T_{\mathbb{P}^1}))$ are naturally deformations of $\pi_{Bl}|_C$ preserving $\pi_{\mathbb{P}^1}|_C$, with the s points of $(\pi_{Bl}|_C)^* E$ required to stay on E.

These are naturally identified with the deformations of π (preserving $\pi_{\mathbb{P}^1}|_C$) where the points of $\pi^{-1} A$ may not move from A. If $\pi : C \to \mathbb{P}^n$ is given by sections $(s_0, \dots, s_n)$ of $\pi^* \mathcal{O}_{\mathbb{P}^n}(1)$ (and A is given by the vanishing of the first two co-ordinates), then these are deformations of $(s_0, \dots, s_n)$ keeping the sections s_0, s_1 constant. Thus the Zariski tangent space to the fiber of σ is isomorphic to $H^0(C, \pi^* \mathcal{O}_{\mathbb{P}^n}(1))^{n-1}$.

The last paragraph can also be rephrased as follows. If $T_{Bl}[-E]$ is the locally free sheaf whose sections are sections of T_{Bl}, and whose sections on E are required to have no normal component, then

$$0 \to (\nu^* \mathcal{O}_{\mathbb{P}^n}(1))^{n-1} \to T_{Bl}[-E] \to m^* T_{\mathbb{P}^1} \to 0$$

is easily seen to be exact. Then pull back to C and take global sections.

(b) Recall that $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^n, d)$ is a smooth stack of dimension $(n+1)(d+1) + m - 4$ (see Appendix A). The Zariski tangent space to the fiber of σ has dimension

$$\begin{aligned} h^0(C, \pi^* \mathcal{O}_{\mathbb{P}^n}(1)^{n-1}) &= (n-1) h^0(C, \pi^* \mathcal{O}_{\mathbb{P}^n}(1)) \\ &= (n-1)(d+1) \\ &= \dim \overline{\mathcal{M}}_{0,m}(\mathbb{P}^n, d) - \dim \overline{\mathcal{M}}_{0,m}(\mathbb{P}^1, d) \end{aligned}$$

as desired.

(c) If no components of C of arithmetic genus 1 are collapsed by $p_A \circ \pi$, then $\rho_A(C, \pi)$ is a smooth point of $\overline{\mathcal{M}}_{1,m}(\mathbb{P}^1, d)$ by Lemma 4.21. If a component B of C of arithmetic genus 1 is collapsed by $p_A \circ \pi$, then the degree of $\pi|_B$ is at least 2 (as C has no genus 1 component contracted by π), so B contains at least 2 (reduced) points of $\pi^{-1} A$. In this case $\rho_A(C, \{p_i\}, \pi)$ consists of a curve with a contracted elliptic component, and this elliptic component has at least two rational tails that map to $\mathbb{P}^1$ with degree 1 (corresponding to the points of $\pi^{-1} A$ on B). Thus by Lemma 4.21, $\rho_A(C, \pi)$ is a smooth point of $\overline{\mathcal{M}}_{1,m}(\mathbb{P}^1, d)$ as well.

By Lemma 4.20, $h^1(C, \pi^*(\mathcal{O}(1))) = 0$, so $h^0(C, \pi^*(\mathcal{O}(1))) = d$ by Riemann-Roch. The proof is then identical to the rest of (b). $\square$

5.7. *Proposition.* — *Fix n, d, $\Gamma = \coprod_{k=0}^{l} \Gamma(k)$, $\Delta = \coprod_{k=0}^{l} \Delta(k)$, and $\mathcal{D}$.*

(a) *$\mathcal{X}_n(d, \Gamma, \Delta)$ has pure dimension*

$$(n+1)d + (n-3) - \sum_{m, \alpha \in S(\Gamma_m)} (n + m - \dim \Gamma_m^\alpha - 2) - \sum_{\alpha \in S(\Delta)} (n - 1 - \dim \Delta^\alpha).$$

The general element of each component is (a map from) a smooth curve. Also, $\mathcal{Y}_n(d(0), \dots, \Delta(l))$ has pure dimension $\dim \mathcal{X}_n(d, \Gamma, \Delta) - 1$.

(b) *$\mathcal{W}_n(d, \Gamma, \Delta)$ has pure dimension*

$$(n+1)d - \sum_{m, \alpha \in S(\Gamma_m)} (n + m - \dim \Gamma_m^\alpha - 2) - \sum_{\alpha \in S(\Delta)} (n - 1 - \dim \Delta^\alpha).$$

The general element of each component is (a map from) a smooth curve. Also, $\mathcal{Y}^a = \mathcal{Y}_n^a(d(0), \dots, \Delta(l))$ (respectively $\mathcal{Y}^b$, $\mathcal{Y}^c$) has pure dimension $\dim \mathcal{W}_n(d, \Gamma, \Delta) - 1$.

(c) *$\mathcal{Z}_n(d, \Delta)_{\mathcal{D}}$ has pure dimension*

$$(n+1)d - \sum_{\alpha \in S(\Delta)} (n - 1 - \dim \Delta^\alpha) - 1.$$

These are the dimensions one would naively expect.

Proof. The proof for (a) is simpler than the proof for (b), and will be omitted for brevity.

(b) We will prove the result about $\dim \mathcal{W}(d, \Gamma, \Delta)$ in the special case $\Delta = \emptyset$ and Γ consists of copies of H. Then the result holds in general by Reductions A and B (Proposition 5.3). In this special case, we must prove that each component of $\mathcal{W}(d, \Gamma, \emptyset)$ is (reduced) of dimension

$$(n+1)d - \sum_m (m-1)|\Gamma_m|.$$

Consider any point $(C, \{p_m^\alpha\}, \pi)$ on $\mathcal{W}(d, \Gamma, \emptyset)$ where no component maps to H and π collapses no component of arithmetic genus 1. The natural map $\mathcal{W}_n(d, \Gamma, \emptyset) \dashrightarrow \mathcal{W}_1(d, \tilde{\Gamma}, \emptyset)$ induced by $\rho_A : \overline{\mathcal{M}}_{1, \sum |\Gamma_m|}(\mathbb{P}^n, d) \dashrightarrow \overline{\mathcal{M}}_{1, \sum |\Gamma_m|}(\mathbb{P}^1, d)$ is smooth of relative dimension $(n-1)d$ at the point $(C, \{p_m^\alpha\}, \pi)$ by Proposition 5.5. The stack $\mathcal{W}_1(d, \tilde{\Gamma}, \emptyset)$ is reduced of dimension $2d + 1 - \sum (m-1)|\Gamma_m|$ by Section 4, so $\mathcal{W}(d, \Gamma, \emptyset)$ has dimension

$$(n-1)d + \dim(\mathcal{W}_1(d, \tilde{\Gamma}, \emptyset)) = (n+1)d - \sum_m (m-1)|\Gamma_m|$$

as desired. As the general element of $\mathcal{W}_1(d, \tilde{\Gamma}, \emptyset)$ is (a map from) a smooth curve, the same is true of $\mathcal{W}(d, \Gamma, \emptyset)$.

The same argument works for $\mathcal{Y}^a$, $\mathcal{Y}^b$, and $\mathcal{Y}^c$, as in Section 4, it was shown that $\mathcal{Y}_1^a(d(0), \dots, \Delta(l))$, $\mathcal{Y}_1^b(d(0), \dots, \Delta(l))$, and $\mathcal{Y}_1^c(d(0), \dots, \Delta(l))$ are Weil divisors of $\mathcal{W}_1(d, \Gamma, \emptyset)$.

(c) By Proposition 5.3 (c), we may assume that Δ consists only of copies of $\mathbb{P}^n$.

It suffices to prove the result for the generically degree $d!$ cover $\mathcal{Z}'_n(d, \Delta)_{\mathcal{D}}$ obtained by marking the points of intersection with a fixed general hyperplane H. If Γ_1 consists of d copies of H, and Γ_m is empty for $m > 1$, then $\mathcal{Z}'_n(d, \Delta)_{\mathcal{D}}$ is a substack of $\mathcal{W}(d, \Gamma, \Delta)$, and as

$$\dim \mathcal{W}_n(d, \Gamma, \Delta) = (n+1)d - \sum_\alpha (n - 1 - \dim \Delta^\alpha),$$

we wish to show that $\mathcal{Z}'_n(d, \Delta)_{\mathcal{D}}$ is a Weil divisor (in fact reduced) of the variety $\mathcal{W}_n(d, \Gamma, \Delta)$.

Assume q^{α_0} appears in $\mathcal{D}$ with non-zero coefficient a (so $\mathcal{D} - aq^{\alpha_0}$ is a sum of integer multiples of q^α, $\alpha \in S(\Delta) \setminus \{\alpha_0\}$). Let $\mathcal{W}(d, \Gamma, \Delta \setminus \{\Delta^{\alpha_0}\})^o$ be the open subset of $\mathcal{W}(d, \Gamma, \Delta \setminus \{\Delta^{\alpha_0}\})$ representing maps from smooth elliptic curves. On the universal curve over $\mathcal{W}(d, \Gamma, \Delta \setminus \{\Delta^{\alpha_0}\})^o$ there is a reduced divisor $\mathcal{Z}$ corresponding to points q such that

$$aq = (\mathcal{D} - aq^\alpha) - \pi^*(\mathcal{O}(1))$$

in the Picard group of the fiber. The universal curve over the stack $\mathcal{W}(d, \Gamma, \Delta \setminus \{\Delta^{\alpha_0}\})$ is $\mathcal{W}(d, \Gamma, \Delta)$ by Proposition 5.3 (a), so by definition the closure of $\mathcal{Z}$ in $\mathcal{W}(d, \Gamma, \Delta)$ is $\mathcal{Z}(d, \Delta)_{\mathcal{D}}$. $\square$

The following proposition is completely irrelevant to the rest of the argument. It is included to ensure that we are actually counting what we might want to.

5.8. *Proposition. — If $n \geq 3$, and $(C, \{p_m^\alpha\}, \{q^\alpha\}, \pi)$ is the stable map corresponding to a general point of a component of $\mathcal{X}(d, \Gamma, \Delta)$ or $\mathcal{W}(d, \Gamma, \Delta)$, then C is smooth and π is a closed immersion.*

Proof. By Reductions A and B (Proposition 5.3), we can assume $\Delta = \emptyset$ and Γ_m consists of copies of H. By the previous proposition, the curve C is smooth. We need only check that π is a closed immersion. The line bundle $\pi^*\mathcal{O}_{\mathbb{P}^n}(d)$ is very ample, so a given non-zero section s_0 and three general sections t_1, t_2, t_3 will separate points and tangent vectors. If $\pi = (s_0, s_1, s_2, s_3, s_4, \dots, s_n)$ then the infinitesimal deformation $(s_0, s_1 + \varepsilon t_1, s_2 + \varepsilon t_2, s_3 + \varepsilon t_3, s_4, \dots, s_n)$ will separate points and tangent vectors and still lie in $\mathcal{X}(d, \Gamma, \Delta)$ (or $\mathcal{W}(d, \Gamma, \Delta)$). As $(C, \{p_m^\alpha\}, \{q^\alpha\}, \pi)$ corresponds to a general point in $\mathcal{X}(d, \Gamma, \Delta)$ (or $\mathcal{W}(d, \Gamma, \Delta)$), the map π must be a closed immersion at this point. $\square$

We will need to avoid the locus on $\mathcal{W}(d, \Gamma, \Delta)$ where an elliptic component is contracted. The following lemma identifies maps which could lie in this locus.

5.9. *Lemma. — Let C be a complete reduced nodal curve of arithmetic genus 1, and let $\pi : C \to \mathbb{P}^n$. Assume (C, π) can be smoothed. If B is a connected union of contracted components of C of arithmetic genus 1, intersecting $\overline{C \setminus B}$ in k points, and $T_1, \dots, T_k$ are the tangent vectors to $\overline{C \setminus B}$ at those points, then the vectors $\{\pi(T_i)\}_{i=1}^k$ in $T_{\pi(B)}\mathbb{P}^n$ are linearly dependent.*

More generally, this result will hold whenever π is a map to an n-dimensional variety X.

Proof. Let Δ be a smooth curve parametrizing maps $(\mathcal{C}_t, \pi)$ (with total family $(\mathcal{C}, \pi)$) to $\mathbb{P}^n$, with $(\mathcal{C}_0, \pi) = (C, \pi)$ and general member a map from a smooth curve. Thus the following diagram commutes.

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\pi} & \mathbb{P}^n \times \Delta \\ & \searrow \quad \swarrow & \\ & \Delta & \end{array}$$

There is an open neighborhood U of $B \subset \mathcal{C}$ such that $\pi \mid_{U \setminus B}$ is an immersion. Thus π factors through a family $\mathcal{C}'$ that is the same as $\mathcal{C}$ except B is contracted. Let π' be the contraction $\pi' : \mathcal{C} \to \mathcal{C}'$. The family $\mathcal{C}'$ is also flat, and its general fiber has genus 1. The central fiber is a union of rational curves, at most nodal away from the image of B. If the images of T_1, $\dots$, T_k in $\mathcal{C}'_0$ are independent, the reduced fiber above 0 would have arithmetic genus 0, so the central fiber (reduced away from the image of B) would have arithmetic genus at most zero, contradicting the constancy of arithmetic genus in flat families. Thus the images of T_1, $\dots$, T_k in $T_{\pi'(B)}\mathcal{C}'_0$ must be dependent, and hence their images in $T_{\pi(B)}\mathbb{P}^n$ must be dependent as well. $\square$

It is likely that every stable map of the form described in the above lemma can be smoothed, which would suggest (via a dimension estimate) that when $k \leq n+1$ those maps with a collapsed elliptic component intersecting k noncontracted components (with linearly dependent images of tangent vectors) form a Weil divisor of $\mathcal{W}(d,\Gamma,\Delta)$. Because of the moduli of $\overline{\mathcal{M}}_{1,k}$, none of these divisors would be enumeratively relevant. Thus the following result is not surprising.

5.10. *Proposition.* — *If $\mathcal{W}'$ is an irreducible substack of $\mathcal{W}(d,\Gamma,\Delta)$ whose general map $(C,\{p_m^\alpha\},\{q^\alpha\},\pi)$ has a contracted elliptic component E (or more generally a contracted connected union of components of arithmetic genus 1) and $\mathcal{W}'$ is of codimension 1, then $\mathcal{W}'$ is enumeratively irrelevant.*

Proof. We show more generally that $\mathcal{W}'$ is stably enumeratively irrelevant (3.5), as this property behaves well with respect to Reductions A and B (Proposition 5.3). By these reductions, we may assume $\Delta = \emptyset$ and Γ consists of copies of H. Restrict to the open substack of $\mathcal{W}$ where the source curve has constant topological type. Take an étale cover to distinguish the components. Let $(C,\{p_m^\alpha\},\pi)$ be a general point of $\mathcal{W}'$. Suppose E has s special points (markings or intersections with noncontracted components) including k intersections with noncontracted components.

If E (with the s special points) has moduli (i.e. the induced map $\mathcal{W}' \to \overline{\mathcal{M}}_{1,s}$ is nonconstant), the family is stably enumeratively irrelevant by Proposition 5.2. Assume that this is not the case.

If $s \geq 3$, replace E by a rational $R = \mathbb{P}^1$, with s fixed special points (i.e. constant in $\overline{\mathcal{M}}_{0,s}$), to obtain a new stable map $(C',\{p_m^\alpha\},\pi') \in \mathcal{X}(d,\Gamma,\emptyset)$. The family of such (C',π') forms an étale cover of a subtack $\mathcal{X}'$ of $\mathcal{X}(d,\Gamma,\emptyset)$, and $\mathcal{X}'$ is contained in $\mathcal{X}''$ where in the latter we don't impose the dependence of tangent vectors required by the previous lemma.

Then $\mathcal{X}''$ has codimension at least $(s-3)+1$ in $\mathcal{X}(d,\Gamma,\emptyset)$ ($s-3$ from the moduli of $\overline{\mathcal{M}}_{0,s}$, 1 from smoothing of the curve). The previous lemma imposes an additional $\max(n+1-k,0)$ conditions, which are independent as the rational curves intersecting R can move freely under automorphisms of $\mathbb{P}^n$ preserving H. Thus the codimension of $\mathcal{X}'$ in $\mathcal{X}(d,\Gamma,\emptyset)$ is at least $n-1+(s-k) \geq n-1$. But $\dim \mathcal{X}(d,\Gamma,\emptyset) - \dim \mathcal{W}(d,\Gamma,\emptyset) = n-3$, so $\dim \mathcal{W}' \leq \dim \mathcal{W}(d,\Gamma,\emptyset) - 2 = \dim \mathcal{W}' - 1$, giving a contradiction.

The $s=2$ case is the same, except E is replaced by nothing. (This is perhaps most quickly said by replacing E by a rational R and then stabilizing.) The same argument carries through.

Otherwise, $k = s = 1$. Then $\mathcal{X}''$ can be identified with the substack of $\mathcal{X}(d,\Gamma,\{\mathbb{P}^n\})$ where the corresponding map $\pi : (C,\{p_m^\alpha\},q) \to \mathbb{P}^n$ is singular at q. As the singularity requirement imposes n conditions,

$$\begin{aligned}
\dim \mathcal{W}' &\leq \dim \mathcal{X}(d,\Gamma,\{\mathbb{P}^n\}) - n \\
&= \dim \mathcal{X}(d,\Gamma,\emptyset) + 1 - n \\
&= (\dim \mathcal{W}(d,\Gamma,\emptyset) + n - 3) + 1 - n \\
&= \dim \mathcal{W}(d,\Gamma,\emptyset) - 2 \\
&= \dim \mathcal{W}' - 1
\end{aligned}$$

giving a contradiction once again. Hence $\mathcal{W}'$ is enumeratively irrelevant. □

6. Components of D_H

Fix d and general linear spaces Γ and Δ (as in the definition of $\mathcal{X}(d,\Gamma,\Delta)$ and $\mathcal{W}(d,\Gamma,\Delta)$). Let q be the marked point corresponding to one of the linear spaces Δ^β in Δ.

Let $D_H = ev_q^* H = \{\pi(q) \in H\}$ be the Cartier divisor on $\mathcal{X}(d, \Gamma, \Delta)$ or $\mathcal{W}(d, \Gamma, \Delta)$ that corresponds to requiring q to lie on H. The components of D_H on $\mathcal{X}(d, \Gamma, \Delta)$ are given by the following result.

6.1. *Theorem. — If Γ and Δ are general, each component of D_H (as a divisor on $\mathcal{X}(d, \Gamma, \Delta)$) is a component of*

$$\mathcal{Y}(d(0), \Gamma(0), \Delta(0); \dots ; d(l), \Gamma(l), \Delta(l))$$

for some l, $d(0)$, ... , $\Delta(l)$, with $d = \sum_{k=0}^{l} d(k)$, $\Gamma = \coprod_{k=0}^{l} \Gamma(k)$, $\Delta = \coprod_{k=0}^{l} \Delta(k)$, $\Delta^\beta \in \Delta(0)$. This component appears with multiplicity $\prod m^k$ (with m^k as defined in Section 3.6).

Proof. By Reductions A and B (Proposition 5.3), we may assume that Γ_m consists only of H's for all m, and that $\Delta = \{\mathbb{P}^n\}$ (with $\Delta^\beta = \mathbb{P}^n$). With these assumptions, the result becomes much simpler. The stack $\mathcal{X}(d, \Gamma, \{\mathbb{P}^n\})$ is the universal curve over $\mathcal{X}(d, \Gamma, \emptyset)$, and we are asking which points of the universal curve lie in $\pi^{-1}H$.

Let $(C, \{p_m^\alpha\}, q, \pi)$ be the stable map corresponding to a general point of a component of D_H. Choose a general $(n-2)$-plane A in H. By Kleiman-Bertini 5.1, the set $\pi^{-1}A$ is a union of reduced points on C, so by Proposition 5.5 ρ_A is smooth (as a morphism of stacks) at the point representing $(C, \{p_m^\alpha\}, q, \pi)$. As a set, D_H contains the entire fiber of ρ_A above $\rho_A(C, \{p_m^\alpha\}, q, \pi)$, so $\rho_A(D_H)$ is a Weil divisor on $\mathcal{X}_1(d, \tilde{\Gamma}, \{\mathbb{P}^1\})$ that is a component of $ev_q^* \infty = \{\pi(q) = \infty\}$ where $\infty = p_A(H)$. By Theorem 4.13, the curve C is a union of irreducible components $C(0) \cup \cdots \cup C(l')$ with $\rho_A \circ \pi(C(0)) = \infty$ (i.e. $\pi(C(0)) \subset H$), $C(0)$ meets $C(k)$, and the marked points split up among the components: $\tilde{\Gamma} = \coprod \tilde{\Gamma}(k)$. If $d(0) = \deg \pi|_{C(0)}$, then $d(0)$ of the curves $C(1)$, ... , $C(l')$ are rational tails that are collapsed to the $d(0)$ points of $C(0) \cap A$; they contain no marked points. Let $l = l' - d(0)$. Also, $\Delta(k) = \emptyset$ for $k > 0$, as the only incidence condition in Δ was $q \in \Delta^\beta$, and $q \in C(0)$.

Therefore this component of D_H is contained in

$$\mathcal{Y} = \mathcal{Y}(d(0), \Gamma(0), \Delta(0); \dots ; d(l), \Gamma(l), \Delta(l)).$$

But $\dim \mathcal{Y} = \dim \mathcal{X}(d, \Gamma, \Delta) - 1$ (by Proposition 5.7), so the set-theoretic result follows. For the multiplicity result, pull back the multiplicity statement of Theorem 4.13 along the smooth morphism ρ_A. $\square$

For enumerative calculations, we need only consider enumeratively relevant components. With this in mind, we restate Theorem 6.1 in language reminiscent of [CH] and [V1]. Let ϕ be the isomorphism of Remark 3.9. The following statement will be more convenient for computation.

6.2. *Theorem. — If Γ and Δ are general, then each enumeratively relevant component of D_H (as a divisor on $\mathcal{X}(d, \Gamma, \Delta)$) is one of the following.*

(I) *A component of $\phi(\mathcal{X}(d, \Gamma', \Delta'))$, where, for some m_0 and $\Delta^{\alpha_0} \in \Gamma_{m_0}$,*
   - *$\Delta' = \Delta \setminus \{\Delta^\beta\}$, and*
   - *Γ' is the same as Γ except $\Gamma_{m_0}^{\alpha_0}$ is replaced by $\Gamma_{m_0}^{\alpha_0} \cap \Delta^\beta$.*

   The component appears with multiplicity m_0. (See Remark 3.9 for the definition of ϕ.)

(II) *A component of $\mathcal{Y}(d(0), \Gamma(0), \Delta(0); \dots ; d(l), \Gamma(l), \Delta(l))$ for some l, $d(0)$, ... , $\Delta(l)$, with $d = \sum_{k=0}^{l} d(k)$, $\Gamma = \coprod_{k=0}^{l} \Gamma(k)$, $\Delta = \coprod_{k=0}^{l} \Delta(k)$, $\Delta^\beta \in \Gamma(0)$, and $d(0) > 0$. The component appears with multiplicity $\prod m^k$ (with m^k as defined in Section 3.6).*

Call these components *Type I components* and *Type II components* respectively.

Proof. Consider a component $\mathcal{Y}$ of D_H that is not a Type II component (so $d(0) = 0$). Let $(C(0) \cup \cdots \cup C(l), \{p_m^\alpha\}, \{q\}, \pi)$ be the stable map corresponding to a general point of $\mathcal{Y}$. The curve $C(0)$ has at least 3 special points: q, one of $\{p_m^\alpha\}$ (call it $p_{m_0}^{\alpha_0}$), and $C(0) \cap C(1)$. If $C(0)$ had more than 3 special points, then the component would not be enumeratively relevant, by Proposition 5.2. Thus $l = 1$, and $\mathcal{Y}$ is a Type I component. $\square$

An analogous result is true in genus 1.

6.3. *Theorem. — If Γ and Δ are general, each enumeratively relevant component of D_H (as a divisor on $\mathcal{W}(d, \Gamma, \Delta)$) is one of the following.*

(I) *A component of $\phi(\mathcal{W}(d, \Gamma', \Delta'))$, where, for some m_0 and $\Delta^{\alpha_0} \in \Gamma_{m_0}$,*
- *$\Delta' = \Delta \setminus \{\Delta^\beta\}$, and*
- *Γ' is the same as Γ except $\Gamma_{m_0}^{\alpha_0}$ is replaced by $\Gamma_{m_0}^{\alpha_0} \cap \Delta^\beta$.*

The component appears with multiplicity m_0.

(II) *A component of $\mathcal{Y}^a(d(0), \dots, \Delta(l))$, $\mathcal{Y}^b(d(0), \dots, \Delta(l))_{m_1^1}$, or $\mathcal{Y}^c(d(0), \dots, \Delta(l))$ for some l, $d(0)$, ... , $\Delta(l)$, with $d = \sum_{k=0}^l d(k)$, $\Gamma = \coprod_{k=0}^l \Gamma(k)$, $\Delta = \coprod_{k=0}^l \Delta(k)$, $\Delta^\beta \in \Gamma(0)$, and $d(0) > 0$. The component appears with multiplicity $\prod m^k$ in cases $\mathcal{Y}^a$ and $\mathcal{Y}^c$, and $m_1^1(m^1 - m_1^1) \prod_{k>1} m^k$ in case $\mathcal{Y}^b$.*

Call the components of (I) *Type I components*, and call the three types of components of (II) *Type IIa, IIb, and IIc components* respectively.

Proof. As in the proof of Theorem 6.1, we assume that Γ consists of copies of H, and $\Delta = \{\mathbb{P}^n\}$ (and $\Delta^\beta = \mathbb{P}^n$). We consider the map $\rho_A : \mathcal{W}_n(d, \Gamma, \{\mathbb{P}^n\}) \dashrightarrow \mathcal{W}_1(d, \tilde{\Gamma}, \{\mathbb{P}^1\})$ near a general point $(C, \{p_m^\alpha\}, \{q\}, \pi)$ of a component of D_H. As a set, D_H contains the entire fiber of ρ_A above $\rho_A(C, \pi)$, so $\rho_A(D_H)$ is a Weil divisor on $\mathcal{W}_1(d, \tilde{\Gamma}, \{\mathbb{P}^1\})$ that is a component of $ev_q^* \infty = \{\pi(q) = \infty\}$. By Theorem 4.14, the curve C is a union of irreducible components $C(0) \cup \dots \cup C(l')$ with $\rho_A \circ \pi(C(0)) = \infty$ (i.e. $C(0) \subset \pi^{-1} H$), $C(0) \cap C(k) \neq \phi$, and the marked points split up among the components: $\tilde{\Gamma} = \coprod \tilde{\Gamma}(k)$. If $d(0) = \deg \pi|_{C(0)}$, then $d(0)$ of the curves $C(1)$, ... , $C(l')$ are rational tails that are collapsed to the $d(0)$ points of $C(0) \cap A$; they contain no marked points. Let $l = l' - d(0)$. Also, $\Delta(k) = \emptyset$ for $k > 0$, as the only incidence condition was $q \in \Delta^\beta$, and $q \in C(0)$.

Case $d(0) > 0$. By Theorem 4.14, the component $\mathcal{Y}$ is contained in

$$\mathcal{Y}^a(d(0), \dots, \Delta(l)), \quad \mathcal{Y}^b(d(0), \dots, \Delta(l)), \quad \text{or} \quad \mathcal{Y}^c(d(0), \dots, \Delta(l)).$$

As the dimensions of each of these three is $\dim \mathcal{W}(d, \Gamma, \Delta) - 1 = \dim \mathcal{Y}$, $\mathcal{Y}$ must be a Type II component as described in the statement of the theorem.

Case $d(0) = 0$. By the same argument as in the proof of Theorem 6.2, $\mathcal{Y}$ is a Type I component.

The multiplicity proof is the same as in Theorem 6.2; pull back the analogous statement in Theorem 4.14 along the smooth morphism ρ_A. $\square$

7. Recursive formulas

7.1. The enumerative geometry of $\mathcal{X}$ and $\mathcal{W}$. Theorems 6.2 and 6.3 express the enumerative geometry of stacks of the form $\mathcal{X}(d(0), \dots, \Delta(l))$ or $\mathcal{W}(d(0), \dots, \Delta(l))$ in terms of the enumerative geometry of stacks of dimension 1 less, of the form $\mathcal{X}$, $\mathcal{W}$, $\mathcal{Y}$, $\mathcal{Y}^a$, $\mathcal{Y}^b$, or $\mathcal{Y}^c$.

7.2. The enumerative geometry of $\mathcal{Y}$ from that of $\mathcal{X}$, and $\mathcal{Y}^a$ and $\mathcal{Y}^c$ from $\mathcal{W}$ and $\mathcal{X}$.

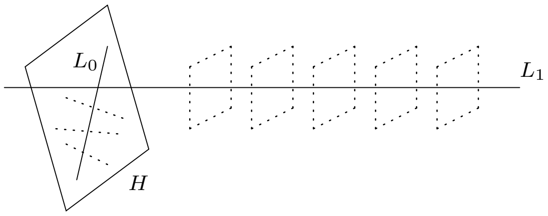

FIGURE 10. How many (L_0, L_1) satisfy the desired conditions?

Counting maps in $\mathcal{Y}(d(0), \dots, \Delta(l))$ (or $\mathcal{Y}^a(d(0), \dots, \Delta(l))$ or $\mathcal{Y}^c(d(0), \dots, \Delta(l))$) corresponds to counting maps of pointed curves $C(0), \dots, C(l)$ where the images of certain points are required to coincide. This is done by the classical method of "splitting the diagonal".

The method is best understood through an example. Fix a hyperplane $H \subset \mathbb{P}^4$. In $\mathbb{P}^4$ the number of ordered pairs of lines (L_0, L_1) consisting of lines $L_0 \subset H$ and $L_1 \subset \mathbb{P}^4$, with L_0 intersecting 3 fixed general lines a_1, a_2, a_3 in H, L_1 intersecting 5 fixed general 2-planes $b_1, \dots, b_5$ in $\mathbb{P}^4$, and L_0 intersecting L_1 (see Figure 10), can be determined as follows.

There is a one-parameter family of lines L_0 in H intersecting the general lines a_1, a_2, a_3. This family sweeps out a surface $S \subset H$ of some degree d_0. The degree d_0 is the number of lines intersecting the lines a_1, a_2, and a_3 *and another general line in* H, so this is

$$\#\mathcal{X}_3(d = 1, \Gamma_1 = \{\text{plane}\}, \Delta = \{a_1, a_2, a_3, \text{another line}\}).$$

There is also a one-parameter family of lines L_1 intersecting the general 2-planes $b_1, \dots, b_5$, and the intersection point of such L_1 with H sweeps out a curve $C \subset H$ of some degree d_1. The degree d_1 is the number of lines intersecting the 2-planes $b_1, \dots, b_5$ in $\mathbb{P}^4$ *and another general 2-plane in* H. Thus

$$d_1 = \#\mathcal{X}_4(d = 1, \Gamma = \{\text{plane in } H\}, \Delta = \{b_1, b_2, b_3, b_4, b_5\}).$$

The answer we seek is $\#(C \cap S) = d_0 d_1$.

We count $\#\mathcal{Y}_n(d(0), \dots, \Delta(l))$ in general as follows. Choose general linear spaces $\gamma(1), \dots, \gamma(l)$ in H of appropriate dimensions such that $\mathcal{X}_n(d(k), \Gamma'(k), \Delta(k))$ is finite, where $\Gamma'(k)$ is $\Gamma(k)$ with the additional condition of $\gamma(k)$ in $\Gamma_{m^k}(k)$. Let $\Gamma'_1(0)$ consist of $d(0)$ copies of a hyperplane H' in $\mathbb{P}^{n-1}$, and $\Gamma'_m(0) = \emptyset$ for $m > 1$. Choose general linear spaces $\delta(1), \dots, \delta(l)$ in H of dimension $\dim \delta(k) = (n-1) - \dim \gamma(k)$ ("dual to $\gamma(k)$"). Let $\Delta'(0) = \Delta(0) \coprod \{\delta(k)\}_{k=1}^{l} \coprod \Gamma(0)$. Then $\mathcal{X}_{n-1}(d(0), \Gamma'(0), \Delta'(0))$ is finite as well. With these definitions,

$$\#\mathcal{Y}_n(d(0), \dots, \Delta(l)) = \frac{\#\mathcal{X}_{n-1}(d(0), \Gamma'(0), \Delta'(0))}{d(0)!} \prod_{k=1}^{l} \#\mathcal{X}_n(d(k), \Gamma'(k), \Delta(k)).$$

The $d(0)!$ is included to account for the possible labellings of the intersection points of a degree $d(0)$ curve $C(0)$ in H with a fixed general hyperplane H' of H.

7.3. We sketch an argument to show that this gives a correct count. If $\Delta''(0)$ is the same as $\Delta(0) \coprod \Gamma(0)$ but with an extra l copies of H, there is a natural map

$$\psi : \mathcal{X}_{n-1}(d(0), \Gamma'_1(0), \Delta''(0)) \to H^l$$

that is the product of the evaluation maps on the "extra l points". If $\Gamma''(k)$ $(k > 0)$ is the same as $\Gamma(k)$ with an additional copy of H in $\Gamma''_{m^k}(k)$, there is a natural map

$$\psi_k : \mathcal{X}_n(d(k), \Gamma''(k), \Delta(k)) \to H$$

that is the evaluation map on the "additional point". As the linear spaces in Γ and Δ are chosen generally, by Kleiman-Bertini 5.1 (with the subgroup of $\mathrm{Aut}(\mathbb{P}^n)$ preserving H, acting on ψ_k), the image of ψ meets the product of the images of ψ_k transversely. The transversality checks for the cases $\mathcal{Y}^a$, $\mathcal{Y}^b$, $\mathcal{Y}^c$ below are similar, and omitted.

7.4. Similar formulas (with the same justification) hold for $\mathcal{Y}^a$ and $\mathcal{Y}^c$.

$$\#\mathcal{Y}^c_n(d(0),\ldots,\Delta(l)) = \#\mathcal{Z}_{n-1}(d(0),\Delta'(0))_{\sum_{m,\alpha} mp^\alpha_m(0)-\sum_{k=1}^l m^k r^k} \prod_{k=1}^l \#\mathcal{X}_n(d(k),\Gamma'(k),\Delta(k))$$

where Γ' and Δ' are defined in the same way as for $\mathcal{Y}$.

$$\begin{aligned}\# \quad & \mathcal{Y}^a_n(d(0),\ldots,\Delta(l)) = \\ & \frac{\#\mathcal{X}_{n-1}(d(0),\Gamma'(0),\Delta'(0))}{d(0)!}\#\mathcal{W}_n(d(1),\Gamma'(1),\Delta(1))\prod_{k=2}^l \#\mathcal{X}_n(d(k),\Gamma'(k),\Delta(k))\end{aligned}$$

where Γ' and Δ' is defined in the same way, except $\gamma(1)$ is chosen so that $\mathcal{W}_n(d(1),\Gamma'(1),\Delta(1))$ is finite.

7.5. The enumerative geometry of $\mathcal{Y}^b$ from that of $\mathcal{X}$. Counting points of $\mathcal{Y}^b$ is similar. For convenience, let $\tilde{\mathcal{Y}}^b$ correspond to the slightly different problem, where we mark one of the points of $C(0)\cap C(1)$ where $C(1)$ intersects H with multiplicity m^1_1; this will avoid a factor of $1/2$ that would otherwise arise if $m^1_1 = m^1_2$. The (mild) additional complexity comes from requiring the curves $C(0)$ and $C(1)$ to intersect twice (at marked points of each curve, on H), so a natural object of study is the blow-up of $H\times H$ along the diagonal Δ, $\mathrm{Bl}_\Delta H\times H$.

But when $n=2$, the situation is simpler. The curve $C(0)$ is H, and $C(1)$ will always intersect it. In this case, for $C(0)\cup\cdots\cup C(l)$ to be determined by the incidence conditions (up to a finite number of possibilities), each of $C(1)$, ... , $C(l)$ must also be determined (up to a finite number). The analogous formula to those of the previous subsection is

$$\#\tilde{\mathcal{Y}}^b_2(d(0);\ldots;\Delta(l))_{m^1_1} = \prod_{k=2}^l \#\mathcal{X}_2(d(k),\Gamma'(k),\Delta(k))$$

where for $k>1$, $\Gamma'(k)$ is the same as $\Gamma(k)$ with the additional condition $\{\text{point}\}$ in $\Gamma_{m^k}(k)$, and $\Gamma'(1)$ is the same as $\Gamma(1)$ with additional conditions $\{\text{point}\}$ in $\Gamma_{m^1_1}(1)$ and $\Gamma_{m^1_2}(1)$. This formula agrees with Theorem 1.3 of [CH].

We now calculate $\#\tilde{\mathcal{Y}}^b$ when $n=3$; the same method works for $n>3$. As an illustration of the method, consider the following enumerative problem.

Fix seven general lines L_1, ... , L_7 in $\mathbb{P}^3$ and a point p on a hyperplane H. How many pairs of curves $(C(0),C(1))$ are there with $C(0)$ a line in H through p and $C(1)$ a conic intersecting L_1, ... , L_7 and intersecting $C(0)$ at two distinct points, where the intersections are labelled a_1 and a_2?

The answer to this enumerative problem is by definition

$$\#\tilde{\mathcal{Y}}^b = \#\tilde{\mathcal{Y}}^b_3(1,\Gamma(0)=\{p\},\emptyset;2,\emptyset,\Delta=\{L_1,\ldots,L_7\}).$$

The space of lines in H passing through p is one-dimensional, and thus defines a three-dimensional locus in $\mathrm{Bl}_\Delta H\times H$. The space of conics in $\mathbb{P}^3$ passing through 7 general lines is one-dimensional, and thus defines a one-dimensional locus in $\mathrm{Bl}_\Delta H\times H$ parametrizing the points of intersection of the conic with H. Then $\#\tilde{\mathcal{Y}}^b$ is the intersection of these two classes.

Let h_i be the class (in the Chow group) of the hyperplane on the i^{th} factor of $\mathrm{Bl}_\Delta H \times H$ $(i = 1, 2)$, and let e be the class of the exceptional divisor. Then the Chow ring of $\mathrm{Bl}_\Delta H \times H$ is generated (as a $\mathbb{Z}$-module) by the classes listed below with the relations

$$h_1^3 = 0, \quad h_2^3 = 0, \quad e^2 = 3h_1 e - h_1^2 - h_1 h_2 - h_2^2.$$

Codimension	Classes
0	1
1	$h_1,\ h_2,\ e$
2	$h_1^2,\ h_1h_2,\ h_2^2,\ h_1e = h_2e$
3	$h_1^2h_2,\ h_1h_2^2,\ h_1^2e = h_1h_2e = h_2^2e$
4	$h_1^2h_2^2$

Let the image of possible pairs of points on $C(0)$ be the class $\mathcal{C}(0) = \alpha(h_1 + h_2) + \beta e$ in $A^3(\mathrm{Bl}_\Delta H \times H)$. Then $\mathcal{C}(0) \cdot h_1^2 h_2 = \alpha$ and $\mathcal{C}(0) \cdot e h_1^2 = -\beta$. But $\mathcal{C}(0) \cdot h_1^2 h_2$ is the number of lines in H passing through p (class $\mathcal{C}(0)$) and another fixed point (class h_1^2) with a marked point on a fixed general line (class h_2), so $\alpha = 1$. Also, $\mathcal{C}(0) \cdot e h_1^2$ is the number of lines in the plane through p (class $\mathcal{C}(0)$) and another fixed point q (class h_1^2) with a marked point mapping to q, so $\beta = -1$. Thus $\mathcal{C}(0) = h_1 + h_2 - e$.

Let the image of possible pairs of points on $C(1) \cap H$ be the class $\mathcal{C}(1) = \gamma(h_1^2 h_2 + h_1 h_2^2) + \delta h_1 h_2 e$. Then $\gamma = \mathcal{C}(1) \cdot h_1$ is the number of conics in $\mathbb{P}^3$ through 7 general lines in $\mathbb{P}^3$ and a general line in H, which is 92 from Section 2.2. Also, $-\delta = \mathcal{C}(1) \cdot e$ counts the number of conics in $\mathbb{P}^3$ through 7 general lines in $\mathbb{P}^3$ and tangent to H, which is 116 (which can be inductively computed using the methods of this article). Thus $\mathcal{C}(1) = 92(h_1^2 h_2 + h_1 h_2^2) - 116 e h_1 h_2$.

Finally, the answer to the enumerative problem is

$$\#\tilde{\mathcal{Y}}^b = (h_1 + h_2 - e)\left(92(h_1^2 h_2 + h_1 h_2^2) - 116 e h_1 h_2\right) = 92 + 92 - 116 = 68.$$

When $n = 3$ in general, there are three cases to consider. Let $C = C(0) \cup C(1) \cup \cdots \cup C(l)$ as usual.

Case i). If the conditions $\Gamma(1)$ and $\Delta(1)$ specify $C(1)$ up to a finite number of possibilities, then $\#\tilde{\mathcal{Y}}^b(d(0), \ldots, \Delta(l))$ is:

$$\#\mathcal{X}(d(1), \Gamma'(1), \Delta(1)) \cdot \#Y(d(0), \Gamma'(0), \Delta(0); d(2), \Gamma(2), \Delta(2); \ldots; d(l), \Gamma(l), \Delta(l))$$

where

- $\Gamma'(1)$ is the same as $\Gamma(1)$ with the additional conditions $\{H\}$ in $\Gamma_{m_1^1}(1)$ and $\Gamma_{m_2^1}(1)$ (the curve $C(1)$ intersects H at two points a_1 and a_2 with multiplicity m_1^1 and m_2^1 respectively; these will be the intersections with $C(0)$), and
- $\Gamma'(0)$ is the same as $\Gamma(0)$ with the additional two conditions $\{\text{point}\}$ and $\{\text{another point}\}$ in $\Gamma'_1(0)$ (the curve $C(0)$ must pass through two points of intersection a_1 and a_2 of $C(1)$ with H).

Case ii). If the conditions $\Gamma(1)$ and $\Delta(1)$ specify $C(1)$ up to a one-parameter family, we are in the same situation as in the enumerative problem above. Then $\#\tilde{\mathcal{Y}}^b(d(0), \ldots; \Delta(l))$ is

$$\Big(d(0)\left(\#\mathcal{X}(d(1), \Gamma', \Delta(1)) + d(0)\#\mathcal{X}(d(1), \Gamma'', \Delta(1))\right)$$

$$-\#\mathcal{X}(d(1), \Gamma''', \Delta(1))\Big) \cdot \#\mathcal{Y}(d(0), \Gamma'(0), \Delta(0); d(2), \ldots, \Delta(l))$$

where

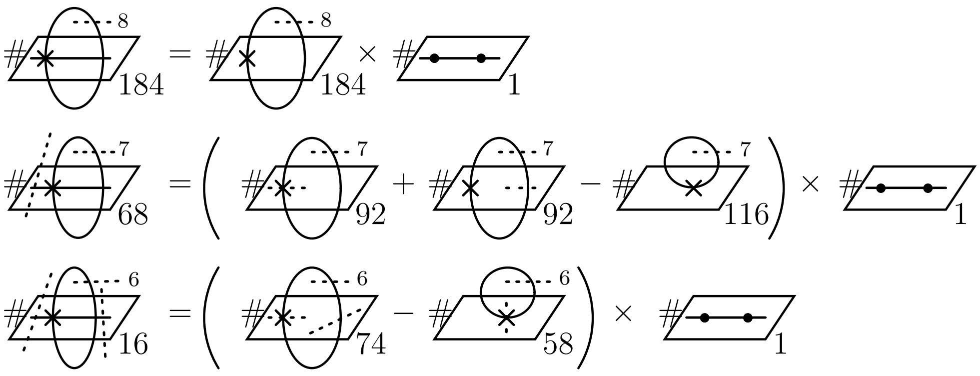

FIGURE 11. Calculating $\#\tilde{\mathcal{Y}}^b$: A pictorial example

- Γ' is the same as $\Gamma(1)$ with additional conditions {line} in $\Gamma'_{m_1^1}$ and $\{H\}$ in $\Gamma'_{m_2^1}$ (the curve $C(1)$ intersects H with multiplicity m_1^1 at a_1 along a fixed general line and with multiplicity m_2^1 at a_2 at another point of H),
- Γ'' is the same as $\Gamma(1)$ with additional conditions $\{H\}$ in $\Gamma''_{m_1^1}$ and {line} in $\Gamma''_{m_2^1}$ (the curve $C(1)$ intersects H with multiplicity m_2^1 at a_2 along a fixed general line and with multiplicity m_1^1 at a_1 at another point of H),
- Γ''' is the same as $\Gamma(1)$ with the additional condition $\{H\}$ in Γ'''_{m^1} (the points a_1 and a_2 on the curve $C(1)$ coincide, and $C(1)$ is required to intersect H at this point with multiplicity $m^1 = m_1^1 + m_2^1$),
- $\Gamma'(0)$ is the same as $\Gamma(0)$ with the additional condition {point} in $\Gamma'_1(0)$ (the curve $C(0)$ is additionally required to pass through a fixed point in H).

Case iii). If the incidence conditions on $C(0) \cup C(2) \cup \cdots \cup C(l)$ specify the union of these curves up to a finite number of possibilities (and the incidence conditions on $C(1)$ specify $C(1)$ up to a two-parameter family), a similar argument shows that $\#\tilde{\mathcal{Y}}^b(d(0), \dots, \Delta(l))$ is

$$(d(0)\#\mathcal{X}(d(1), \Gamma'(1), \Delta(1)) - \#\mathcal{X}(d(1), \Gamma''(1), \Delta(1)))\, \#Y(d(0), \Gamma(0), \Delta(0); d(2), \dots, \Delta(l))$$

where

- $\Gamma'(1)$ is the same as $\Gamma(1)$ with the additional conditions {line} in $\Gamma'_{m_1^1}(1)$ and {another line} in $\Gamma'_{m_2^1}(1)$
- $\Gamma''(1)$ is the same as $\Gamma(1)$ with the additional condition {line} in $\Gamma''_{m^1}(1)$.

These three cases are illustrated pictorially in Figure 11 for the special case of conics in $\mathbb{P}^3$ intersecting a line in H at two points, with the entire configuration required to intersect 8 general lines in $\mathbb{P}^3$. One of the intersection points of the conic with H is marked with an "$\times$" to remind the reader of the marking a_1. The distribution of the line conditions (e.g. the number of line conditions on the conic) is indicated by a small number. The bigger number beside each picture is the actual solution to the enumerative problem corresponding to the picture. For example, there are 116 conics in $\mathbb{P}^3$ tangent to a general hyperplane H intersecting 7 general lines.

7.6. In $\mathbb{P}^n$ $(n > 3)$, the procedure is the same, but the Chow ring of $\mathrm{Bl}_\Delta(H \times H)$ is more complicated, so there are more cases.

7.7. The enumerative geometry of $\mathcal{Z}$ from that of $\mathcal{X}$ and $\mathcal{W}$. For the purposes of this section, ${}^d\Gamma$ is defined by: ${}^d\Gamma_1$ contains of d copies of H, and ${}^d\Gamma_k = \emptyset$ for $k > 1$.

We will use intersection theory on elliptic fibrations over a curve; the Chow ring modulo algebraic or numerical equivalence will suffice. Let $\mathcal{F}$ be an elliptic fibration over a smooth curve with smooth total space, whose fibers are smooth elliptic curves, except for a finite number of fibers which are irreducible nodal elliptic curves. Let F be the class of a fiber and $K_{\mathcal{F}}$ the (class of the) relative dualizing sheaf. The self-intersection of a section is independent of the choice of section. (*Proof:* $K_{\mathcal{F}}$ restricted to the generic fiber is trivial, so $K_{\mathcal{F}}$ is a sum of fibers. Let S_1, S_2 be two sections. Using adjunction, $S_1^2 + K_{\mathcal{F}} \cdot S_1 = (K_{\mathcal{F}} + S_1) \cdot S_1 = 0$, so $S_1^2 = -K_{\mathcal{F}} \cdot S_1 = -K_{\mathcal{F}} \cdot S_2 = S_2^2$.)

For convenience, call the self-intersection of a section S^2. The parenthetical proof above shows that $K_{\mathcal{F}} = -S^2 F$.

7.8. *Proposition.* — *Let S be a section, and C a class on $\mathcal{F}$ such that $S = C$ on the general fiber. Then $S = C + (\frac{S^2 - C^2}{2})F$.*

Proof. As all fibers are irreducible, $S = C + kF$ for some k. By adjunction,

$$0 = S \cdot (K_{\mathcal{F}} + S) = (C + kF)(C + (k - S^2)F) = C^2 + 2k - S^2.$$

Hence $k = (S^2 - C^2)/2$. □

If the dimension of $\mathcal{Z}_n(d, \Delta)_{\sum m^\beta q^\beta}$ is 0, then consider the universal family over the curve parametrizing maps to $\mathbb{P}^n$ with only the incidence conditions of Δ (i.e. no divisorial condition). This is $\mathcal{W}_n(d, {}^d\Gamma, \Delta)$ modulo the symmetric group S_d (to forget the d markings of intersection with H). The general point of $\mathcal{W}_n(d, {}^d\Gamma, \Delta)$ represents a smooth elliptic curve. The remaining points represent curves that are either irreducible and rational, or elliptic with rational tails. (This can be proved by simple dimension counts on $\mathcal{W}_1(d, {}^d\Gamma, \Delta)$.) Normalize the base (which will normalize the family), and blow down (-1)-curves in fibers.

Call the resulting family $\mathcal{F}$. It is straightforward to check that the total space of $\mathcal{F}$ is smooth. (To check that the surface is smooth at the nodes of the family, show that the first-order deformations of such maps surject onto the deformation space of the node, using the long exact deformation sequence of Appendix A and the fact that $H^1(C, \pi^* T_X) = 0$. By appropriate application of Kleiman-Bertini 5.1, one checks that the point remains smooth once one requires marked points to lie on various general linear spaces.)

The curves blown down come from maps from nodal curves $C(0) \cup C(1)$, where $C(0)$ is rational and $C(1)$ is elliptic. Let H be the pullback of a hyperplane to $\mathcal{F}$, and let Q^β be the section given by q^β.

7.9. *Theorem.* — *Let $D = H - \sum m^\beta Q^\beta$. Then $\#\mathcal{Z}_n(d, \Delta)_{\sum m^\beta q^\beta} = S^2 - D^2/2$.*

Proof. Let Q be any section. Let S be the section given by $Q + H - \sum m^\beta q^\beta$ in the Picard group of the generic fiber. Then

$$\#\mathcal{Z}_n(d, \Delta)_{\sum m^\beta q^\beta} = S \cdot Q.$$

(The sections S and Q intersect transversely from the proof of Proposition 5.7(c).) By the previous proposition, as $S^2 = Q^2$,

$$\begin{aligned} S &= Q + D + \left(\frac{S^2 - (Q+D)^2}{2} \right) F \\ \text{so } S \cdot Q &= \left(Q + D + \left(\frac{S^2 - (Q+D)^2}{2} \right) F \right) \cdot Q \\ &= Q^2 + D \cdot Q + \frac{S^2 - Q^2 - D^2}{2} - D \cdot Q \\ &= S^2 - D^2/2. \end{aligned}$$

□

To calculate

$$\#\mathcal{Z}_n(d, \Delta)_{\sum m^\beta q^\beta} = S^2 - (H - \sum m^\beta Q^\beta)^2/2,$$

we need to calculate H^2, $H \cdot Q^\beta$, and $Q^\beta \cdot Q^{\beta'}$, and these correspond to simpler enumerative problems.

If $\beta \neq \beta'$, Q^β could intersect $Q^{\beta'}$ in two ways. If $\dim \Delta^\beta + \dim \Delta^{\beta'} \geq n$, the elliptic curve could pass through $\Delta^\beta \cap \Delta^{\beta'}$, which will happen

$$\#\mathcal{W}(d, {}^d\Gamma, \Delta \setminus \{\Delta^\beta, \Delta^{\beta'}\} \coprod \{\Delta^\beta \cap \Delta^{\beta'}\})/d! \tag{6}$$

times. Or the curve could break into two intersecting components, one rational containing Q^β and $Q^{\beta'}$ (which will be blown down in the construction of $\mathcal{F}$, with images of Q^β and $Q^{\beta'}$ meeting transversely), and the other smooth elliptic. This will happen

$$\sum_{\substack{d(0)+d(1)=d \\ \Delta(0) \coprod \Delta(1) = \Delta \\ \Delta^\beta, \Delta^{\beta'}}} (d(0)d(1))^{\delta_{n,2}} \left(\frac{\#\mathcal{X}(d(0), {}^{d(0)}\Gamma, \Delta(0))}{d(0)!} \right) \left(\frac{\#\mathcal{W}(d(1), {}^{d(1)}\Gamma, \Delta(1))}{d(1)!} \right) \tag{7}$$

times. The factor of $(d(0)d(1))^{\delta_{n,2}}$ corresponds to the fact when $n = 2$, $\pi(C)$ is a plane curve, and the node of C could map to any node of the plane curve $\pi(C)$. Transversality in both cases is simple to check, and both possibilities are of the right dimension. Thus $Q^\beta \cdot Q^{\beta'}$ is the sum of (6) and (7).

To determine $H \cdot Q^\beta$ on $\mathcal{F}$, fix a general hyperplane h in $\mathbb{P}^n$, and let H be its pullback to the fibration $\mathcal{F}$. Then H is a multisection of the elliptic fibration. The cycle H could intersect Q^β in two ways. Either $\pi(q^\beta) \in h \cap \Delta^\beta$ — which will happen

$$\#W(d, {}^d\Gamma, \Delta \setminus \{\Delta^\beta\} \coprod \{h \cap \Delta^\beta\})/d! \tag{8}$$

times — or the curve breaks into two pieces, one rational containing a point of h and q^β, which will happen

$$\sum_{\substack{d(0)+d(1)=d \\ \Delta(0) \coprod \Delta(1) = \Delta \\ \Delta^\beta \in \Delta(0)}} (d(0)d(1))^{\delta_{n,2}} d(0) \left(\frac{\#\mathcal{X}(d(0), {}^{d(0)}\Gamma, \Delta(0))}{d(0)!} \right) \left(\frac{\#\mathcal{W}(d(1), {}^{d(1)}\Gamma, \Delta(1))}{d(1)!} \right) \tag{9}$$

times. (The second $d(0)$ in the formula comes from the choice of point of h on the degree $d(0)$ rational component.) Thus $H \cdot Q^\beta$ is the sum of (8) and (9).

To determine H^2, fix a second general hyperplane h' in $\mathbb{P}^n$, and let H' be its pullback to $\mathcal{F}$. Once again, H could intersect H' in two ways depending on if the curve passes through $h \cap h'$, or if the curve breaks into two pieces. The first case happens

$$\frac{\#\mathcal{W}(d, {}^d\Gamma, \Delta \coprod \{h \cap h'\})}{d!} \tag{10}$$

times, and the second happens

$$\sum_{\substack{d(0)+d(1)=d \\ \Delta(0) \coprod \Delta(1) = \Delta}} (d(0)d(1))^{\delta_{n,2}} d(0)^2 \left(\frac{\#\mathcal{X}(d(0), {}^{d(0)}\Gamma, \Delta(0))}{d(0)!} \right) \left(\frac{\#\mathcal{W}(d(1), {}^{d(1)}\Gamma, \Delta(1))}{d(1)!} \right) \tag{11}$$

times. Hence H^2 is the sum of (10) and (11).

The self-intersection of a section S^2 $(= (Q^\beta)^2)$ can be calculated as follows. We can calculate $H \cdot Q^\beta$, so if we can evaluate $(H - Q^\beta) \cdot Q^\beta$ then we can find $S^2 = (Q^\beta)^2$. Fix a general hyperplane h containing Δ^β, and let $(H - Q^\beta)$ be the multisection that is the pullback of h to $\mathcal{F}$, minus the section Q^β. The cycle $(H - Q^\beta)$ intersects Q^β if the curve is tangent to h along Δ^β, or if the curve breaks into two pieces, with Q^β on the rational piece. The first case happens

$$\frac{\#\mathcal{W}(d, \Gamma', \Delta \setminus \{\Delta^\beta\})}{(d-2)!} \tag{12}$$

times, where Γ'_1 contains $d-2$ copies of h and Γ'_2 contains one copy of Δ^β (which is contained in h). The second case happens

$$\sum_{\substack{d(0)+d(1)=d, d(0) \geq 2 \\ \Delta(0) \coprod \Delta(1) = \Delta \setminus \Delta^\beta}} (d(0)d(1))^{\delta_{n,2}} \left(\frac{\#\mathcal{X}(d(0), \Gamma(0), \Delta(0))}{(d(0)-2)!} \right) \left(\frac{\#\mathcal{W}(d(1), {}^{d(1)}\Gamma, \Delta(1))}{d(1)!} \right) \tag{13}$$

times, where $\Gamma(0)$ consists of $(d(0)-1)$ copies of h and one copy of Δ^β. Hence $(H - Q^\beta) \cdot Q^\beta$ is the sum of (12) and (13). The denominator $(d(0)-2)!$ arises because we have a degree $d(0)$ (rational) curve passing through a linear space Δ^β on h, and various incidence conditions $\Delta(0)$. The number of such curves with a choice of one of the other intersections of $C(0)$ with h is $(d(0)-1)\#\mathcal{X}(d(0), \Gamma(0), \Delta(0))/(d(0)-1)!$.

As an example, consider the elliptic quartics in $\mathbb{P}^2$ passing through 11 fixed points, including q^1, q^2, q^3, q^4. How many such two-nodal quartics have $\mathcal{O}(1) = q^1 + \cdots + q^4$ in the Picard group of the normalization of the curve? We construct the fibration $\mathcal{F}$ over the (normalized) variety of two-nodal plane quartics through 11 fixed points. We have sections $Q^1, \ldots, Q^{11}$ and a multisection H. If $\beta \neq \beta'$, $Q^\beta \cdot Q^{\beta'} = 3$, $H \cdot Q^\beta = 30$, $H^2 = 225 + 3\binom{11}{2} = 390$, $(H - Q^1) \cdot Q^1 = 185$, so

$$S^2 = H \cdot Q^1 - (H - Q^1) \cdot Q^1 = -155.$$

Let $D = H - Q^1 - Q^2 - Q^3 - Q^4$. Then

$$D^2 = H^2 + 4S^2 - 8H \cdot Q^1 + 12 Q^1 \cdot Q^2 = 390 + 4(-155) - 8(30) + 12(3) = -434$$

so the answer is $S^2 - D^2/2 = 62$.

To determine the enumerative geometry of quartic elliptic space curves (see Section 8.3), various $\#\mathcal{Z}_2(d, \Delta)_{\mathcal{D}}$ were needed with $d = 3$ and $d = 4$. When $d = 3$, Δ consists of 8 points (for $\dim \mathcal{Z}$ to be 0) and some lines; the results are given in the Table 1. For convenience, we write p^β for the base points (in Δ) and l^β for the marked points on lines. These values were independently confirmed by M. Roth ([Ro]). When $d = 4$, Δ consists of 11 points and some lines; the results are given in the Table 2. For convenience again, we write p^β for points and l^β for lines.

$\#l^\beta$	$\mathcal{D}$	$\#\mathcal{Z}_2(d,\Delta)_\mathcal{D}$
0	$p^1+p^2+p^3$	0
1	$p^1+p^2+l^1$	1
2	$p^1+l^1+l^2$	5
3	$l^1+l^2+l^3$	18
0	p^1+2p^2	1
1	$2p^1+l^1$	4
1	p^1+2l^1	5
2	l^1+2l^2	16
0	$3p^1$	3
1	$3l^1$	14
0	$p^1+p^2+p^3+p^4-p^5$	1
1	$p^1+p^2+p^3+p^4-l^1$	2
1	$p^1+p^2+p^3+l^1-p^4$	4
2	$p^1+p^2+p^3+l^1-l^2$	10
2	$p^1+p^2+l^1+l^2-p^3$	14
3	$p^1+p^2+l^1+l^2-l^3$	39
3	$p^1+l^1+l^2+l^3-p^2$	45
4	$p^1+l^1+l^2+l^3-l^4$	135
4	$l^1+l^2+l^3+l^4-p^1$	135
5	$l^1+l^2+l^3+l^4-l^5$	432

TABLE 1. Counting cubic elliptic plane curves with a divisorial condition

$\#l^\beta$	$\mathcal{D}$	$\#\mathcal{Z}_2(d,\Delta)_\mathcal{D}$
0	$p^1+p^2+p^3+p^4$	62
1	$p^1+p^2+p^3+l^1$	464
2	$p^1+p^2+l^1+l^2$	2,522
3	$p^1+l^1+l^2+l^3$	11,960
4	$l^1+l^2+l^3+l^4$	52,160

TABLE 2. Counting quartic elliptic plane curves with a divisorial condition

8. EXAMPLES

8.1. Plane curves. Type IIc components in this case are never enumeratively relevant, as the elliptic curve $C(0)$ must map to the line H with degree at least two. The recursive formulas we get are identical to the genus 1 recursive formulas of Caporaso and Harris in [CH].

8.2. Cubic elliptic space curves. The number of smooth cubic elliptic space curves through j general points and $12-2j$ general lines is 1500, 150, 14, and 1 for $j=0$, 1, 2, and 3 respectively. (The number is 0 for $j>3$ as cubic elliptic space curves must lie in a plane.) The degenerations involved in calculating the first case appeared in Section 2.4. As the Chow ring of the space of smooth elliptic cubics is not hard to calculate (see [H] p. 36), these results may be easily verified.

The number of cubic elliptics tangent to H, through j general points and $11-2j$ general lines is 4740, 498, 50, and 4 for $j=0$, 1, 2, and 3 respectively. The number of cubic elliptics triply tangent to H through j general points and $10-2j$ general lines is 2790, 306, 33, and 3 for $j=0$, 1, 2, and 3 respectively.

j	# quartics
0	52,832,040
1	4,436,208
2	385,656
3	34,674
4	3,220
5	310
6	32
7	4
8	1

TABLE 3. Number of quartic elliptic space curves through j general points and $16-2j$ general lines

These numbers are needed for the next examples.

8.3. Quartic elliptic space curves. The number of smooth quartic elliptic space curves through j general points and $16-2j$ general lines is given in the Table 3. These numbers agree with those independently found by Getzler ([G], proof to appear in [GP]).

Other enumerative data can also be found en route. For example, Tables 4 and 5 give the number of smooth quartic elliptic space curves through i_0 general points and i_1 general lines, and h_0 general points and h_1 general lines in H, with

$$2i_0 + i_1 + 2h_0 + h_1 = 16.$$

At each stage, the number may be computed by degenerating a point or a line (assuming there is a point or line to degenerate). Each row is labelled, and the labels of the different degenerations that are also smooth quartics are given in each case, and a "+" is added if there are other degenerations (where the curve breaks into pieces). (This will help the reader to follow through the degenerations.) Keep in mind that these numbers are not quite what the algorithm of this article produces; in the algorithm, the intersections with H are labelled, so the number computed for (i_0, i_1, h_0, h_1) will be $(4-h_0-h_1)!$ times the number in the table.

These computations are not as difficult as one might think (and were done by hand). For example, if i_0 and i_1 are both positive, it is possible to degenerate a point and then a line, or a line and then a point. Both methods must yield the same number, providing a means of double-checking.

As an example, some of the degenerations used to compute the 52,832,040 quartic space curves through 16 general lines are displayed in Figure 12, using the pictorial shorthand of Section 2. Degenerations involving irreducible quartic elliptic space curves are given (as well as a few more). The boldfaced numbers indicate the corresponding rows in Tables 4 and 5. The phrase "etc." indicates that further degenerations appear. A full diagram of degenerations is available on request.

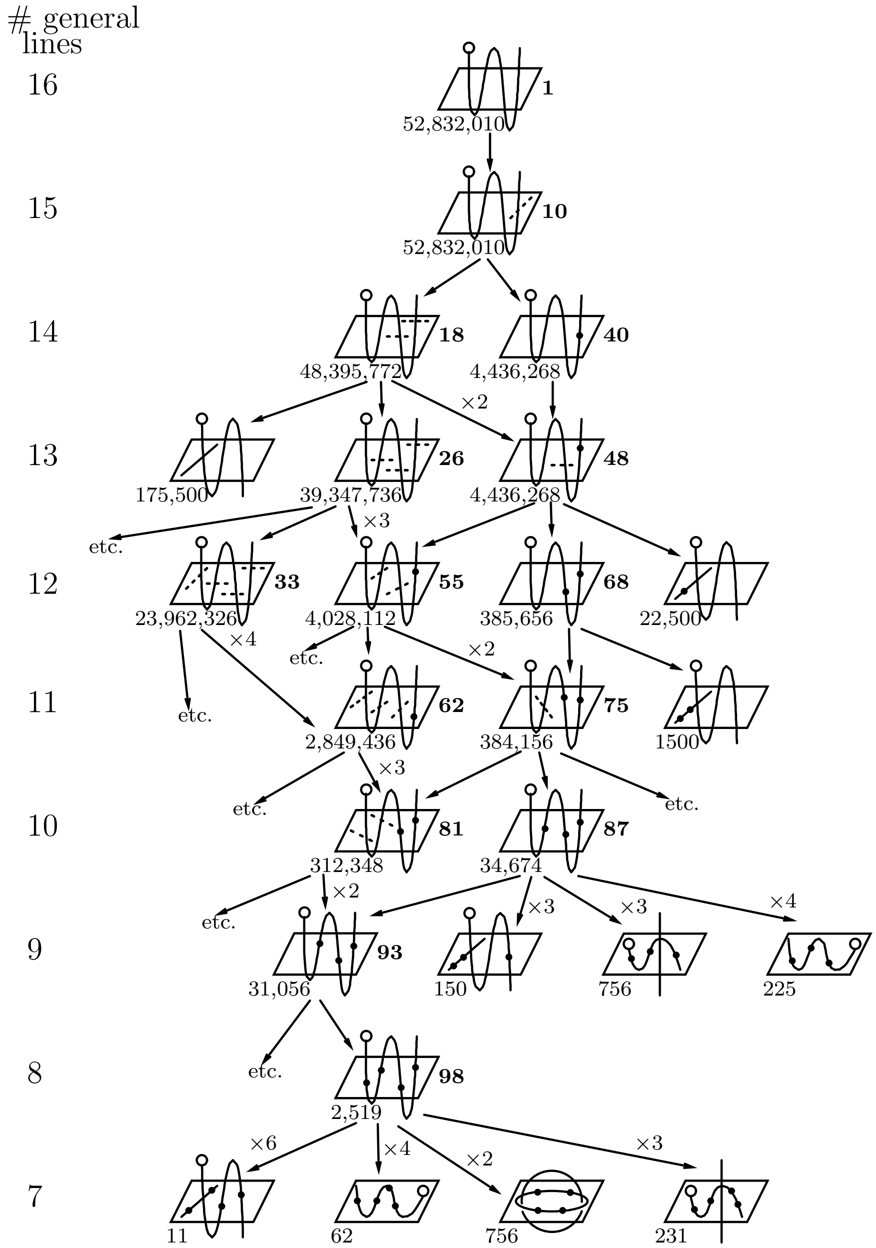

FIGURE 12. Counting quartic elliptic space curves through 16 general lines

	$(i_0, i_1,$ $h_0, h_1)$	point degen.	line degen.	# curves
1	(16,0,0,0)		10	52,832,040
2	(14,1,0,0)	40	11	4,436,268
3	(12,2,0,0)	41	12	385,656
4	(10,3,0,0)	42	13	34,674
5	(8,4,0,0)	43	14	3,220
6	(6,5,0,0)	44	15	310
7	(4,6,0,0)	45	16	32
8	(1,7,0,0)	46	17	4
9	(0,8,0,0)	47		1
10	(15,0,1,0)		18, 40	52,832,040
11	(13,1,1,0)	48	19, 41	4,436,268
12	(11,2,1,0)	49	20, 42	385,656
13	(9,3,1,0)	50	21, 43	34,674
14	(7,4,1,0)	51	22, 44	3,220
15	(5,5,1,0)	52	23, 45	310
16	(3,6,1,0)	53	24, 46	32
17	(1,7,1,0)	54	25, 47	4
18	(14,0,2,0)		26, 48+	48,395,772
19	(12,1,2,0)	55+	27, 49+	4,050,612
20	(10,2,2,0)	56+	28, 50+	350,982
21	(8,3,2,0)	57+	29, 51+	31,454
22	(6,4,2,0)	58+	30, 52	2,910
23	(4,5,2,0)	59	31, 53	278
24	(2,6,2,0)	60	32, 54	28
25	(0,7,2,0)	61		3
26	(13,0,3,0)		33, 55+	39,347,736
27	(11,1,3,0)	62+	34, 56+	3,266,100
28	(9,2,3,0)	63+	35, 57+	280,752
29	(7,3,3,0)	64+	36, 58+	24,972
30	(5,4,3,0)	65+	37, 59+	2,290
31	(3,5,3,0)	66+	38, 60+	214
32	(1,6,3,0)	67+	39, 61+	20
33	(12,0,4,0)		62+	23,962,326
34	(10,1,4,0)	+	63+	1,939,857
35	(8,2,4,0)	+	64+	161,735
36	(6,3,4,0)	+	65+	13,908
37	(4,4,4,0)	+	66+	1,222
38	(2,5,4,0)	+	67+	104
39	(0,6,4,0)	+		8
40	(14,0,0,1)		48+	4,436,268
41	(12,1,0,1)	68	49	385,656
42	(10,2,0,1)	69	50	34,674
43	(8,3,0,1)	70	51	3,220
44	(6,4,0,1)	71	52	310
45	(3,5,0,1)	72	53	32
46	(2,6,0,1)	73	54	4
47	(0,7,0,1)	74		1
48	(13,0,1,1)		55, 68+	4,436,268
49	(11,1,1,1)	75+	56, 69+	385,656
50	(9,2,1,1)	76+	57, 70+	34,674
51	(7,3,1,1)	77+	58, 71+	3,220

Table 4. Quartic elliptic space curves with incidence conditions

	$(i_0, i_1,$ $h_0, h_1)$	point degen.	line degen.	# curves
52	(5,4,1,1)	78+	59, 72	310
53	(3,5,1,1)	79	60, 73	32
54	(1,6,1,1)	80	61, 74	4
55	(12,0,2,1)		62, 75	4,028,112
56	(10,1,2,1)	81+	63, 76+	349,032
57	(8,2,2,1)	82+	64, 77+	28,340
58	(6,3,2,1)	83+	65, 78+	2,901
59	(4,4,2,1)	84+	66, 79+	278
60	(2,5,2,1)	85+	67, 80+	28
61	(0,6,2,1)	86+		3
62	(11,0,3,1)		81+	2,849,436
63	(9,1,3,1)	+	82+	243,507
64	(7,2,3,1)	+	83+	21,310
65	(5,3,3,1)	+	84+	1,909
66	(3,4,3,1)	+	85+	172
67	(1,5,3,1)	+	86+	14
68	(12,0,0,2)		75+	385,656
69	(10,1,0,2)	87	76+	34,674
70	(8,2,0,2)	88	77+	3,220
71	(6,3,0,2)	89	78+	310
72	(4,4,0,2)	90	79	32
73	(2,5,0,2)	91	80	4
74	(0,6,0,2)	92		1
75	(11,0,1,2)		81, 87+	384,156
76	(9,1,1,2)	93+	82, 88+	34,524
77	(7,2,1,2)	94+	83, 89+	3,206
78	(5,3,1,2)	95+	84, 90+	309
79	(3,4,1,2)	96	85, 91+	32
80	(1,5,1,2)	97	86, 92+	4
81	(10,0,2,2)		93+	312,348
82	(8,1,2,2)	+	94+	28,340
83	(6,2,2,2)	+	95+	2,612
84	(4,3,2,2)	+	96+	246
85	(2,4,2,2)	+	97+	24
86	(0,5,2,2)	+		2
87	(10,0,0,3)		93+	34,674
88	(8,1,0,3)	98+	94+	3,220
89	(6,2,0,3)	99+	95+	310
90	(4,3,0,3)	100+	96	32
91	(2,4,0,3)	101	97	4
92	(0,5,0,3)	102		1
93	(9,0,1,3)		98+	31,056
94	(7,1,1,3)	+	99+	3,052
95	(5,2,1,3)	+	100+	304
96	(3,3,1,3)	+	101+	32
97	(1,4,1,3)	+	102+	4
98	(8,0,0,4)		+	2,519
99	(6,1,0,4)	+	+	277
100	(4,2,0,4)	+	+	31
101	(2,3,0,4)	+	+	4
102	(0,4,0,4)	+		1

Table 5. Quartic elliptic space curves with incidence conditions, cont'd

Appendix A. Background: The moduli space of stable maps

For the convenience of the reader we recall certain facts about moduli stacks of stable maps to smooth varieties, without proofs.

A *family of m-pointed nodal curves over a base scheme S* (or a *nodal curve over S*) is a proper flat morphism $\rho : C \to S$ whose geometric fibers are reduced and pure dimension 1, with at worst ordinary double points as singularities, with sections $\sigma_i : S \to C$ $(1 \le i \le m)$ whose images are disjoint, and lie in the smooth locus of ρ. If X is a scheme, then a *family of maps of nodal curves to X over S* (or a *map of a nodal curve to X over S*) is a morphism $\pi : C \to X \times S$ of schemes over S, where the induced morphism $\rho : C \to S$ is a family of nodal curves over S. A *nodal curve* (with no base scheme specified) is a nodal curve over $\operatorname{Spec} \mathbb{C}$, and a *map of a nodal curve to X* is a map over $\operatorname{Spec} \mathbb{C}$. Similar definitions hold for families of nodal curves (and maps) over Deligne-Mumford stacks (see [DMu] for definitions).

A *stable map* is a map π from a connected pointed nodal curve to a smooth variety X such that π has finite automorphism group. The *arithmetic genus* of a stable map is defined to be the arithmetic genus of the nodal curve C. If $[C] \in H_2(C)$ is the fundamental class of C, and $\beta = \pi_*[C] \in H_2(X)$, then we say the image of C is in class β. If X is a projective space, and β is d times the class of a line, we say that d is the *degree* of the stable map. The requirement that the automorphism group be finite is equivalent to requiring each collapsed genus 0 component to have at least three special points and each contracted genus 1 component to have at least one special point, where a special point is defined to be either a marked point or a branch of a node.

A *family of stable maps to X* is a family of maps of connected pointed nodal curves to X whose fibers over closed points are stable maps. Let $\overline{\mathcal{M}}_{g,m}(X,\beta)$ be the stack whose category of sections of a scheme S is the category of families of stable maps to X over S of arithmetic genus g with m marked points, with image in class β. If X is a smooth projective variety, $\overline{\mathcal{M}}_{g,m}(X,\beta)$ is a fine moduli stack of Deligne-Mumford type. There is an open substack $\mathcal{M}_g(X,\beta)$ (possibly empty) that is a fine moduli stack of maps of *smooth* curves.

The stack $\overline{\mathcal{M}}_{g,m+1}(X,\beta)$ is the universal curve over $\overline{\mathcal{M}}_{g,m}(X,\beta)$. A morphism of smooth projective varieties $f : X \to Y$ induces a morphism of stacks $\overline{\mathcal{M}}_{g,m}(X, f^*\beta) \to \overline{\mathcal{M}}_{g,m}(Y,\beta)$ ([BM], Remark after Theorem 3.14). There are natural *evaluation maps* $ev_i : \overline{\mathcal{M}}_{g,m}(X,\beta) \to X$ $(1 \le i \le m)$ that informally give "the image of the ith marked point".

The versal deformation space to the map $(C, \{p_i\}, \pi)$ in $\overline{\mathcal{M}}_{g,m}(X,\beta)$ is obtained from the complex

$$\underline{\Omega}_\pi = \left(\pi^*\Omega_X \to \Omega_C(p_1 + \cdots + p_m)\right).$$

The vector space $\mathbb{H}\mathrm{om}(\underline{\Omega}_\pi, \mathcal{O}_C)$ parametrizes infinitesimal automorphisms of the map $(C, \{p_i\}, \pi)$ (denoted $\operatorname{Aut}(C, \{p_i\}, \pi)$); if $(C, \{p_i\}, \pi)$ is a stable map, $\mathbb{H}\mathrm{om}(\underline{\Omega}_\pi, \mathcal{O}_C) = 0$. The space of infinitesimal deformations to the map $(C, \{p_i\}, \pi)$ (i.e. the Zariski tangent space to $\overline{\mathcal{M}}_{g,m}(X,\beta)$ at the point representing this stable map, denoted $\operatorname{Def}(C, \{p_i\}, \pi)$) is given by $\mathbb{E}\mathrm{xt}^1(\underline{\Omega}_\pi, \mathcal{O}_C)$ and the obstruction space (denoted $\operatorname{Ob}_{(C,\{p_i\},\pi)}$) is given by $\mathbb{E}\mathrm{xt}^2(\underline{\Omega}_\pi, \mathcal{O}_C)$.

By applying the functor $\operatorname{Hom}(\cdot, \mathcal{O}_C)$ to the exact sequence of complexes

$$0 \to \Omega_C(p_1 + \cdots + p_m) \to \underline{\Omega}_\pi \to \pi^*\Omega_X[1] \to 0$$

we obtain the long exact sequence

$$\begin{aligned}
0 \to \mathbb{H}\mathrm{om}(\underline{\Omega}_\pi, \mathcal{O}_C) &\to \operatorname{Hom}(\Omega_C(p_1 + \cdots + p_m), \mathcal{O}_C) \to H^0(C, \pi^*T_X) \\
\to \mathbb{E}\mathrm{xt}^1(\underline{\Omega}_\pi, \mathcal{O}_C) &\to \operatorname{Ext}^1(\Omega_C(p_1 + \cdots + p_m), \mathcal{O}_C) \to H^1(C, \pi^*T_X) \\
\to \mathbb{E}\mathrm{xt}^2(\underline{\Omega}_\pi, \mathcal{O}_C) &\to 0.
\end{aligned}$$

By the identifications given in the previous paragraph, and using $\mathrm{Hom}(\Omega_C(p_1+\cdots+p_m),\mathcal{O}_C) = \mathrm{Aut}(C,\{p_i\})$ and $\mathrm{Ext}^1(\Omega_C(p_1+\cdots+p_m),\mathcal{O}_C) = \mathrm{Def}(C,\{p_i\})$, this long exact sequence can be rewritten as

$$\begin{array}{rcl} 0 & \longrightarrow & \mathrm{Aut}(C,\{p_i\}) \longrightarrow H^0(C,\pi^*T_X) \\ \longrightarrow \mathrm{Def}(C,\{p_i\},\pi) & \longrightarrow & \mathrm{Def}(C,\{p_i\}) \longrightarrow H^1(C,\pi^*T_X) \\ \longrightarrow \mathrm{Ob}(C,\{p_i\},\pi) & \longrightarrow & 0. \end{array}$$

The term $H^0(C,\pi^*T_X)$ can be interpreted as first order deformations of the map π, with the pointed curve $(C,\{p_i\})$ fixed.

If $H^1(C,\pi^*T_X) = 0$ at a point $(C,\{p_i\},\pi)$, then the obstruction space vanishes, and the moduli stack is smooth. Thus $\overline{\mathcal{M}}_{0,m}(\mathbb{P}^n,d)$ is smooth of dimension $(n+1)(d+1)+m-4$, and $\overline{\mathcal{M}}_{1,m}(\mathbb{P}^n,d)$ is smooth of dimension $(n+1)d+m-1$ at points $(C,\{p_i\},\pi)$ where $d>1$ and C is smooth.

The construction of the versal deformation space from $\underline{\Omega}_\pi$ is discussed in [Ra], [Vi], and [LT]. All other facts described here appear in [BM] or in the comprehensive introduction [FP].

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