$\overline{M}_g$ IS IRREDUCIBLE

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April 1998: The original note was from some time in 1996. I've edited it slightly, and removed excessively naive (or wrong) statements.

We will show that $\overline{M}_g$ is irreducible (in characteristic 0) using semistable reduction and minimal facts about $\overline{M}_g$. This idea was provoked by a comment Joe Harris once made about the power of tightly-controlled codimension 1 degenerations. I have since looked at Fulton's two-page note "On the Irreducibility of the Moduli Space of Curves", and I realize that this note is only the additional observation that we don't even need to invoke anything like the compactification of the Hurwitz scheme.

0.1. **A degeneration question.** Fix a point $\infty \in \mathbb{P}^1$, and consider a regular curve C with a degree d map π to $\mathbb{P}^1$, with r simple ramifications away from ∞, and $\pi^{-1}(\infty)$ a union of p distinct points. Call these p points "∞-sections". Then by Riemann-Hurwitz, we have

$$r = d + 2g + p - 2 \qquad (1)$$

ramification points away from ∞. Move one of the ramification points to ∞, *keeping the others fixed.* We can use the (characteristic 0) recipe for semistable reduction to find a limit map from a nodal curve, after an appropriate base change.[1] (Essentially, take the limit stable map.) By base-changing at the start, we may assume that the ∞-sections are distinguishable.

The limit curve has two parts, one part C' consisting of components mapping dominantly to $\mathbb{P}^1$, and the other C^∞ of components mapping

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[1] In a nutshell, take any limit map — where the total family of curves if flat, and there is a family of maps, but the central fiber may have non-reduced components and various singularities. Blow up the surface until it is regular and the central fiber is set-theoretically nodal, make a base change of order the *lcm* of the multiplicities of the components of the central fiber, normalize, and then blow down (-1)-curves (of the family) on the central fiber that don't map dominantly to $\mathbb{P}^1$.

to ∞. By blowing up further, we may assume C' is regular, and that the total space of the family is regular (so the ∞-sections have limits that are regular points of the central fiber). Let g' and g^∞ be the arithmetic genera of C', C^∞ respectively, n the number of nodes where C' and C^∞ meet, p' the number of preimages of ∞ on C', β the number of the p ∞-sections whose limit is in C^∞. Of the p' pre-images of ∞ in C', n of them are points of intersection with C^∞, and the rest are limits of the remaining $p - \beta$ ∞-sections of the general curve (which could theoretically come together). Hence

$$p' \leq n + (p - \beta). \tag{2}$$

By Riemann-Hurwitz for the curve C' (similar to (1)), $r - 1 = d + 2g' + p' - 2$, so (comparing with (1))

$$2g' + p' = 2g + p - 1. \tag{3}$$

As the arithmetic genus of the central fiber is g,

$$g' + g^\infty + n - 1 = g. \tag{4}$$

By combining (2)-(4) ((2) - (3)+ 2 (4)),

$$2g^\infty - 2 + n + \beta \leq 1. \tag{5}$$

For the j^{th} of the (say, k) connected components of C^∞, let g_j^∞ be the arithmetic genus, let n_j be the number of intersections with C', and let β_j be the number of (limits of) ∞-sections on it. Then (5) can be restated as $\sum_{j=1}^k (2g_j^\infty - 2 + n_j + \beta_j) \leq 1$. As the central fiber is connected, $n_j \geq 1$. Also, for each connected component of C^∞, at least one of the p ∞-sections must lie on it.[2] Hence $\sum_{j=1}^k 2g_j^\infty \leq 1$, so the arithmetic genus of each connected component of C^k must be 0. Thus all collapsed components of the central fiber are rational. (We can conclude more, but we won't need to for our purposes.)

0.2. **Brief sketch of irreducibility argument.** By the usual arguments, we need only show that any regular genus g curve C can be

[2] This is intuitively clear to me, but I haven't thought of a two-line argument, although I'm sure one exists. Here's a longer argument. Assume otherwise that C_j^∞ is a connected component of C^∞ not meeting any ∞-section. The pullback of $\mathcal{O}_{\mathbb{P}^1}(\infty)$ to the universal family has degree 0 when restricted to C_j^∞. The pullback of the divisor ∞ is a positive linear combination of irreducible components of C_j^∞ (plus other components not meeting C_j^∞). But each irreducible component of C_j^∞ has non-positive intersection number with C_j^∞, and at least one has strictly negative intersection (as $(C_j^\infty)^2 = -n_j$). Thus we have a contradiction.

degenerated to a nodal stable curve. (By this, I mean that the component of $\overline{M}_g$ containing $[C]$ also contains a nodal curve.) Map C to $\mathbb{P}^1$ so that all ramification is simple. Fix a point $\infty \in \mathbb{P}^1$. Specialize the ramifications to lie over ∞ one at a time. If it breaks into pieces where the stable model has a genus g component, this component must map dominantly to $\mathbb{P}^1$, so we'll throw away the "rational tails" and continue. (This may decrease d.) This process can't continue forever, as no dominant morphism from a genus g curve to $\mathbb{P}^1$ can be ramified over only one point.