THE MODULI SPACE OF CURVES AND GROMOV-WITTEN THEORY

RAVI VAKIL

ABSTRACT. The goal of this article is to motivate and describe how Gromov-Witten theory can and has provided tools to understand the moduli space of curves. For example, ideas and methods from Gromov-Witten theory have led to both conjectures and theorems showing that the tautological part of the cohomology ring has a remarkable and profound structure. As an illustration, we describe a new approach to Faber's intersection number conjecture via branched covers of the projective line (work with I.P. Goulden and D.M. Jackson, based on work with T. Graber). En route we review the work of a large number of mathematicians.

CONTENTS

1. INTRODUCTION

These notes are intended to explain how Gromov-Witten theory has been useful in understanding the moduli space of complex curves. We will focus on the moduli space of smooth curves and how much of the recent progress in understanding it has come through "enumerative" invariants in Gromov-Witten theory, something which we take

Date: May 26, 2006.
Partially supported by NSF CAREER/PECASE Grant DMS–0228011, and an Alfred P. Sloan Research Fellowship.
2000 Mathematics Subject Classification: Primary 14H10, 14H81, 14N35, Secondary 14N10, 53D45, 14H15.

for granted these days, but which should really be seen as surprising. There is one sense in which it should not be surprising — in many circumstances, modern arguments can be loosely interpreted as the fact that we can understand curves in general by studying branched covers of the complex projective line, as all curves can be so expressed. We will see this theme throughout the notes, from a Riemann-style parameter count in §2.2 to the tool of relative virtual localization in Gromov Witten theory in §5.

These notes culminate in an approach to Faber's intersection number conjecture using relative Gromov-Witten theory (joint work with Goulden and Jackson [GJV3]). One motivation for this article is to convince the reader that our approach is natural and straightforward.

We first introduce the *moduli space of curves*, both the moduli space of smooth curves, and the Deligne-Mumford compactification, which we will see is something forced upon us by nature, not arbitrarily imposed by man. We will then define certain geometrically natural cohomology classes on the moduli space of smooth curves (the tautological subring of the cohomology ring), and discuss Faber's foundational conjectures on this subring. We will then extend these notions to the moduli space of stable curves, and discuss Faber-type conjectures in this context. A key example is Witten's conjecture, which really preceded (and motivated) Faber's conjectures, and opened the floodgates to the last decade's flurry of developments. We will then discuss other relations in the tautological ring (both known and conjectural). We will describe Theorem $\star$ (Theorem 4.1), a blunt tool for proving many statements, and Y.-P. Lee's Invariance Conjecture, which may give all relations in the tautological ring. In order to discuss the proof of Theorem $\star$, we will be finally drawn into Gromov-Witten theory, and we will quickly review the necessary background. In particular, we will need the notion of "relative Gromov-Witten theory", including Jun Li's degeneration formula [Li1, Li2] and the relative virtual localization formula [GrV3]. Finally, we will use these ideas to tackle Faber's intersection number conjecture.

Because the audience has a diverse background, this article is intended to be read at many different levels, with as much rigor as the reader is able to bring to it. Unless the reader has a solid knowledge of the foundations of algebraic geometry, which is most likely not the case, he or she will have to be willing to take a few notions on faith, and to ask a local expert a few questions.

We will cover a lot of ground, but hopefully this article will include enough background that the reader can make explicit computations to see that he or she can actively manipulate the ideas involved. You are strongly encouraged to try these ideas out via the exercises. They are of varying difficulty, and the amount of rigor required for their solution should depend on your background.

Here are some suggestions for further reading. For a gentle and quick introduction to the moduli space of curves and its tautological ring, see [V2]. For a pleasant and very detailed discussion of moduli of curves, see Harris and Morrison's foundational book [HM]. An on-line resource discussing curves and links to topology (including a glossary of important terms) is available at [GiaM]. For more on curves, Gromov-Witten theory, and localization, see [HKKPTVVZ, Chapter 22–27], which is intended for both physicists and

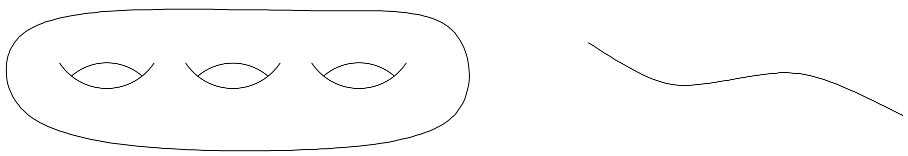

FIGURE 1. A complex curve, and its real "cartoon"

mathematicians. Cox and Katz' wonderful book [CK] gives an excellent mathematical approach to mirror symmetry. There is as of yet no ideal book introducing (Deligne-Mumford) stacks, but Fantechi's [Fan] and Edidin's [E] both give an excellent idea of how to think about them and work with them, and the appendix to Vistoli's paper [Vi] lays out the foundations directly, elegantly, and quickly, although this is necessarily a more serious read.

Acknowledgments. I am grateful to the organizers of the June 2005 conference in Cetraro, Italy on "Enumerative invariants in algebraic geometry and string theory" (Kai Behrend, Barbara Fantechi, and Marco Manetti), and to Fondazione C.I.M.E. (Centro Internazionale Matematico Estivo). I learned this material from my co-authors Graber, Goulden, and Jackson, and from the other experts in the field, including Carel Faber, Rahul Pandharipande, Y.-P. Lee, ..., whose names are mentioned throughout this article. I thank Carel Faber, Soren Galatius, Tom Graber, Arthur Greenspoon, Y.-P. Lee and Rahul Pandharipande for improving the manuscript. I am very grateful to Renzo Cavalieri, Sam Payne, and the participants in their April 2006 "Moduli space of Curves and Gromov-Witten Theory Workshop" at the University of Michigan for their close reading and many detailed suggestions.

2. THE MODULI SPACE OF CURVES

We begin with some conventions and terminology. We will work over $\mathbb{C}$, although these questions remain interesting over arbitrary fields. We will work algebraically, and hence only briefly mention other important approaches to the subjects, such as the construction of the moduli space of curves as a quotient of Teichmüller space.

By *smooth curve*, we mean a compact (also known as proper or complete), smooth (also known as nonsingular) complex curve, i.e. a Riemann surface, see Figure 1. Our curves will be connected unless we especially describe them as "possibly disconnected". In general our *dimensions* will be algebraic or complex, which is why we refer to a Riemann surface as a curve — they have algebraic/complex dimension 1. Algebraic geometers tend to draw "half-dimensional" cartoons of curves (see also Figure 1).

The reader likely needs no motivation to be interested in Riemann surfaces. A natural question when you first hear of such objects is: what are the Riemann surfaces? How many of them are there? In other words, this question asks for a classification of curves.

2.1. *Genus.* A first invariant is the *genus* of the smooth curve, which can be interpreted in three ways: (i) the number of holes (*topological genus*; for example, the genus of the curve in Figure 1 is 3), (ii) dimension of space of differentials ($= h^0(C, \Omega_C)$, *geometric genus*), and (iii) the first cohomology group of the sheaf of algebraic functions ($h^1(C, \mathcal{O}_C)$, *arithmetic genus*). These three notions are the same. Notions (ii) and (iii) are related by *Serre duality*

$$\boxed{H^0(C, \mathcal{F}) \times H^1(C, \mathcal{K} \otimes \mathcal{F}^*) \to H^1(C, \mathcal{K}) \cong \mathbb{C}} \tag{1}$$

where $\mathcal{K}$ is the canonical line bundle, which for smooth curves is the sheaf of differentials Ω_C. Here $\mathcal{F}$ can be any finite rank vector bundle; H^i refers to sheaf cohomology. Serre duality implies that $h^0(C, \mathcal{F}) = h^1(C, \mathcal{K} \otimes \mathcal{F}^*)$, hence (taking $\mathcal{F} = \mathcal{K}$). $h^0(C, \Omega_C) = h^1(C, \mathcal{O}_C)$. (We will use these important facts in the future!)

As we are working purely algebraically, we will not discuss why (i) is the same as (ii) and (iii).

2.2. There is a $(3g - 3)$-dimensional family of genus g curves.

Remarkably, it was already known to Riemann [R, p. 134] that there is a "$(3g - 3)$-dimensional family of genus g curves". You will notice that this can't possibly be right if $g = 0$, and you may know that this isn't right if $g = 1$, as you may have heard that elliptic curves are parametrized by the j-line, which is one-dimensional. So we will take $g > 1$, although there is a way to extend to $g = 0$ and $g = 1$ by making general enough definitions. (Thus there is a "(-3)-dimensional moduli space" of genus 0 curves, if you define moduli space appropriately — in this case as an Artin stack. But that is another story.)

Let us now convince ourselves (informally) that there is a $(3g - 3)$-dimensional family of genus g curves. This will give me a chance to introduce some useful facts that we will use later. I will use the same notation for vector bundles and their sheaves of sections. The sheaf of sections of a line bundle is called an *invertible sheaf*.

We will use five ingredients.

(1) *Serre duality* (1). This is a hard fact.

(2) *The Riemann-Roch formula.* If $\mathcal{F}$ is any coherent sheaf (for example, a finite rank vector bundle) then

$$\boxed{h^0(C, \mathcal{F}) - h^1(C, \mathcal{F}) = \deg \mathcal{F} - g + 1.}$$

This is an easy fact, although I will not explain why it is true.

(3) Line bundles of negative degree have no non-zero sections: if $\mathcal{L}$ is a line bundle of negative degree, then $\boxed{h^0(C, \mathcal{L}) = 0}$. Here is why: the degree of a line bundle $\mathcal{L}$ can be defined as follows. Let s be any non-zero meromorphic section of $\mathcal{L}$. Then the degree of $\mathcal{L}$ is the number of zeros of s minus the number of poles of s. Thus if $\mathcal{L}$ has an honest non-zero section (with no poles), then the degree of s is at least 0.

Exercise. If $\mathcal{L}$ is a degree 0 line bundle with a non-zero section s, show that $\mathcal{L}$ is isomorphic to the trivial bundle (the sheaf of functions) $\mathcal{O}$.

(4) Hence if $\mathcal{L}$ is a line bundle with $\deg \mathcal{L} > \deg \mathcal{K}$, then $h^1(C, \mathcal{L}) = 0$ by Serre duality, from which $\boxed{h^0(C, \mathcal{L}) = \deg \mathcal{L} - g + 1}$ by Riemann-Roch.

(5) *The Riemann-Hurwitz formula.* Suppose $C \to \mathbb{P}^1$ is a degree d cover of the complex projective line by a genus g curve C, with ramification $r_1, \dots, r_n$ at the ramification points on C. Then

$$\boxed{\chi_{\mathrm{top}}(C) = d\chi_{\mathrm{top}}(\mathbb{P}^1) - \sum (r_i - 1),}$$

where χ_{top} is the topological Euler characteristic, i.e.

$$\boxed{2 - 2g = 2d - \sum (r_i - 1).} \tag{2}$$

We quickly review the language of divisors and line bundles on smooth curves. A *divisor* is a formal linear combination of points on C, with integer coefficients, finitely many non-zero. A divisor is *effective* if the coefficients are non-negative. The *degree* of a divisor is the sum of its coefficients. Given a divisor $D = \sum n_i p_i$ (where the p_i form a finite set), we obtain a line bundle $\mathcal{O}(D)$ by "twisting the trivial bundle n_i times at the point p_i". This is best understood in terms of the sheaf of sections. Sections of the sheaf $\mathcal{O}(D)$ (over some open set) correspond to meromorphic functions that are holomorphic away from the p_i; and if $n_i > 0$, have a pole of order at most n_i at p_i; and if $n_i < 0$, have a zero of order at least $-n_i$ at p_i. Each divisor yields a line bundle along with a meromorphic section (obtained by taking the function 1 in the previous sentence's description). Conversely, each line bundle with a non-zero meromorphic section yields a divisor, by taking the "divisor of zeros and poles": if s is a non-zero meromorphic section, we take the divisor which is the sum of the zeros of s (with multiplicity) minus the sum of the poles of s (with multiplicity). These two constructions are inverse to each other. In short, line bundles with the additional data of a non-zero *meromorphic* section correspond to divisors. This identification is actually quite subtle the first few times you see it, and it is worth thinking through it carefully if you have not done so before. Similarly, line bundles with the additional data of a non-zero *holomorphic* section correspond to *effective* divisors.

We now begin our dimension count. We do it in three steps.

Step 1. Fix a curve C, and a degree d. Let $\operatorname{Pic}^d C$ be the set of degree d line bundles on C. Pick a point $p \in C$. Then there is an bijection $\operatorname{Pic}^0 C \to \operatorname{Pic}^d C$ given by $\mathcal{F} \to \mathcal{F}(dp)$. (By $\mathcal{F}(dp)$, we mean the "twist of $\mathcal{F}$ at p, d times", which is the same construction sketched two paragraphs previously. In terms of sheaves, if $d > 0$, this means the sheaf of meromorphic sections of $\mathcal{F}$, that are required to be holomorphic away from p, but may have a pole of order at most d at p. If $d < 0$, this means the sheaf of holomorphic sections of $\mathcal{F}$ that are required to have a zero of order at least $-d$ at p.) If we believe $\operatorname{Pic}^d C$ has some nice structure, which is indeed the case, then we would expect that this would be an isomorphism. In fact, Pic^d can be given the structure of a complex manifold or complex variety, and this gives an isomorphism of manifolds or varieties.

Step 2: "$\dim \operatorname{Pic}^d C = g$." There are quotes around this equation because so far, $\operatorname{Pic}^d C$ is simply a set, so this will just be a plausibility argument. By Step 1, it suffices to consider any $d > \deg \mathcal{K}$. Say $\dim \operatorname{Pic}^d C = h$. We ask: how many degree d *effective divisors* are there (i.e. what is the dimension of this family)? The answer is clearly d, and C^d surjects onto this set (and is usually $d!$-to-1).

But we can count effective divisors in a different way. There is an h-dimensional family of line bundles by hypothesis, and each one of these has a $(d-g+1)$-dimensional family of non-zero sections, each of which gives a divisor of zeros. But two sections yield the same divisor if one is a multiple of the other. Hence we get: $h+(d-g+1)-1 = h+d-g$.

Thus $d = h+d-g$, from which $h = g$ as desired.

Note that we get a bit more: if we believe that Pic^d has an algebraic structure, we have a fibration $(C^d)/S_d \to \operatorname{Pic}^d$, where the fibers are isomorphic to $\mathbb{P}^{d-g}$. In particular, Pic^d is reduced (I won't define this!), and irreducible. (In fact, as many of you know, it is isomorphic to the dimension g abelian variety $\operatorname{Pic}^0 C$.)

Step 3. Say $\mathcal{M}_g$ has dimension p. By fact **(4)** above, if $d \gg 0$, and D is a divisor of degree d, then $h^0(C, \mathcal{O}(D)) = d-g+1$. If we take two general sections s, t of the line bundle $\mathcal{O}(D)$, we get a map to $\mathbb{P}^1$ (given by $p \to [s(p); t(p)]$ — note that this is well-defined), and this map is degree d (the preimage of $[0;1]$ is precisely $\operatorname{div} s$, which has d points counted with multiplicity). Conversely, any degree d cover $f : C \to \mathbb{P}^1$ arises from two linearly independent sections of a degree d line bundle. (To get the divisor associated to one of them, consider $f^{-1}([0;1])$, where points are counted with multiplicities; to get the divisor associated to the other, consider $f^{-1}([1;0])$.) Note that (s,t) gives the same map to $\mathbb{P}^1$ as (s',t') if and only (s,t) is a scalar multiple of (s',t'). Hence the number of maps to $\mathbb{P}^1$ arising from a fixed curve C and a fixed line bundle $\mathcal{L}$ corresponds to the choices of two sections ($2(d-g+1)$ by fact **(4)**), minus 1 to forget the scalar multiple, for a total of $2d-2g+1$. If we let the the line bundle vary, the number of maps from a fixed curve is $2d-2g+1+\dim \operatorname{Pic}^d(C) = 2d-g+1$. If we let the curve also vary, we see that the number of degree d genus g covers of $\mathbb{P}^1$ is $\boxed{p+2d-g+1}$.

But we can also count this number using the Riemann-Hurwitz formula (2). By that formula, there will be a total of $2g+2d-2$ branch points (including multiplicity). Given the branch points (again, with multiplicity), there is a finite amount of possible monodromy data around the branch points. The Riemann Existence Theorem tells us that given any such monodromy data, we can uniquely reconstruct the cover, so we have

$$p+2d-g+1 = 2g+2d-2,$$

from which $\boxed{p = 3g-3}$.

Thus there is a $(3g-3)$-dimensional family of genus g curves! (By showing that the space of branched covers is reduced and irreducible, we could again "show" that the moduli space is reduced and irreducible.)

2.3. The moduli space of smooth curves.

It is time to actually define the moduli space of genus g smooth curves, denoted $\mathcal{M}_g$, or at least to come close to it. By "moduli space of curves" we mean a "parameter space for curves". As a first approximation, we mean the set of curves, but we want to endow this set with further structure (ideally that of a manifold, or even of a smooth complex variety). This structure should be given by nature, not arbitrarily defined.

Certainly if there were such a space $\mathcal{M}_g$, we would expect a universal curve over it $\mathcal{C}_g \to \mathcal{M}_g$, so that the fiber above the point $[C]$ representing a curve C would be that same C. Moreover, whenever we had a family of curves parametrized by some base B, say $\mathcal{C}_B \to B$ (where the fiber above any point $b \in B$ is some smooth genus g curve C_b), there should be a map $f : B \to \mathcal{M}_g$ (at the level of sets sending $b \in B$ to $[C_b] \in \mathcal{M}_g$), and then $f^*\mathcal{C}_g$ should be isomorphic to $\mathcal{C}_B$.

We can turn this into a precise definition. The families we should consider should be "nice" ("fibrations" in the sense of differential geometry). It turns out that the corresponding algebraic notion of "nice" is *flat*, which I will not define here. We can *define* $\mathcal{M}_g$ to be the scheme such that the maps from any scheme B to it are in natural bijection with nice (flat) families of genus g curves over B. (Henceforth all families will be assumed to be "nice"=flat.) Some thought will convince you that only one space (up to isomorphism) exists with this property. This "abstract nonsense" is called *Yoneda's Lemma*. The argument is general, and applies to nice families of any sort of thing. Categorical translation: we are saying that this contravariant functor of families is *represented* by the functor $\mathrm{Hom}(\cdot, \mathcal{M}_g)$. Translation: if such a space exists, then it is unique, up to unique isomorphism.

If there is such a moduli space $\mathcal{M}_g$, we gain some additional information: cohomology classes on $\mathcal{M}_g$ are "characteristic classes" for families of genus g curves. More precisely, given any family of genus g curves $\mathcal{C}_B \to B$, and any cohomology class $\alpha \in H^*(\mathcal{M}_g)$, we have a cohomology class on B: if $f : B \to \mathcal{M}_g$ is the moduli map, take $f^*\alpha$. These characteristic classes behave well with respect to pullback: if $\mathcal{C}_{B'} \to B'$ is a family obtained by pullback from $\mathcal{C}_B \to B$, then the cohomology class on B' induced by α is the pullback of the cohomology class on B induced by α. The converse turns out to be true: any such "universal cohomology class", defined for all families and well-behaved under pullback, arises from a cohomology class on $\mathcal{M}_g$. (The argument is actually quite tautological, and the reader is invited to think it through.) More generally, statements about the geometry of $\mathcal{M}_g$ correspond to "universal statements about all families".

Here is an example of a consequence. A curve is *hyperelliptic* if it admits a 2-to-1 cover of $\mathbb{P}^1$. In the space of smooth genus 3 curves $\mathcal{M}_3$, there is a Cartier divisor of hyperelliptic curves, which means that the locus of hyperelliptic curves is locally cut out by a single equation. Hence in *any* family of genus 3 curves over an arbitrarily horrible base, the hyperelliptic locus are cut out by a single equation. (For scheme-theoretic experts: for any family $\mathcal{C}_B \to B$ of genus 3 curves, there is then a closed subscheme of B corresponding to the hyperelliptic locus. What is an intrinsic scheme-theoretic definition of this locus?)

Hence all we have to do is show that there is such a scheme $\mathcal{M}_g$. Sadly, there is no such scheme! We could just throw up our hands and end these notes here. There are two patches to this problem. One solution is to relax the definition of moduli space (to get the notion of *coarse moduli space*), which doesn't quite parametrize all families of curves. A second option is to extend the notion of *space*. The first choice is the more traditional one, but it is becoming increasingly clear that the second choice the better one.

This leads us to the notion of a *stack*, or in this case, the especially nice stack known as a *Deligne-Mumford stack*. This is an extension of the idea of a scheme. Defining a Deligne-Mumford stack correctly takes some time, and is rather tiring and uninspiring, but dealing with Deligne-Mumford stacks on a day-to-day basis is not so bad — you just pretend it is a scheme. One might compare it to driving a car without knowing how the engine works, but really it is more like driving a car while having only the vaguest idea of what a car is.

Thus I will content myself with giving you a few cautions about where your informal notion of Deligne-Mumford stack should differ with your notion of scheme. (I feel less guilty about this knowing that many analytic readers will be similarly uncomfortable with the notion of a scheme.) The main issue is that when considering cohomology rings (or the algebraic analog, Chow rings), we will take $\mathbb{Q}$-coefficients in order to avoid subtle technical issues. The foundations of intersection theory for Deligne-Mumford stacks were laid by Vistoli in [Vi] (However, thanks to work of Andrew Kresch [Kr], it is possible to take integral coefficients using the Chow ring. Then we have to accept the fact that cohomology groups can be non-zero even in degree higher than the dimension of the space. This is actually something that for various reasons we *want* to be true, but such a discussion is not appropriate in these notes.)

A smooth (or nonsingular) Deligne-Mumford stack (over $\mathbb{C}$) is essentially the same thing as a complex orbifold. The main caution about saying that they are the same thing is that there are actually three different definitions of orbifold in use, and many users are convinced that their version is the only version in use, causing confusion for readers such as myself.

Hence for the rest of these notes, we will take for granted that there is a moduli space of smooth curves $\mathcal{M}_g$ (and we will make similar assumptions about other moduli spaces).

Here are some *facts* about the moduli space of curves. The space $\mathcal{M}_g$ has (complex) dimension $3g-3$. It is smooth (as a stack), so it is an orbifold (given the appropriate definition), and we will imagine that it is a manifold. We have informally seen that it is irreducible.

We make a brief excursion outside of algebraic geometry to show that this space has some interesting structure. In the analytic setting, $\mathcal{M}_g$ can be expressed as the quotient of *Teichmüller space* (a subset of $\mathbb{C}^{3g-3}$ homeomorphic to a ball) by a discrete group, known as the *mapping class group*. Hence the cohomology of the quotient $\mathcal{M}_g$ is the group cohomology of the mapping class group. (Here it is essential that we take the quotient as an orbifold/stack.) Here is a fact suggesting that the topology of this space has some elegant structure:

$$\chi(\mathcal{M}_g) = B_{2g}/2g(2g-2) \tag{3}$$

(due to Harer and Zagier [HZ]), where B_{2g} denotes the $2g$th Bernoulli number.

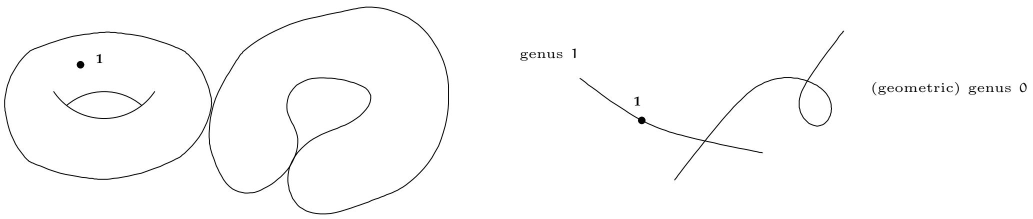

FIGURE 2. A pointed nodal curve, and its real "cartoon"

Other exciting recent work showing the attractive structure of the cohomology ring is Madsen and Weiss' proof of Madsen's generalization of Mumford's conjecture [MW]. We briefly give the statement. There is a natural isomorphism between $H^*(\mathcal{M}_g;\mathbb{Q})$ and $H^*(\mathcal{M}_{g+1};\mathbb{Q})$ for $* < (g-1)/2$ (due to Harer and Ivanov). Hence we can define the ring we could informally denote by $H^*(\mathcal{M}_\infty;\mathbb{Q})$. Mumford conjectured that this is a free polynomial ring generated by certain cohomology classes (κ-classes, to be defined in §3.1). Madsen and Weiss proved this, and a good deal more. (See [T] for an overview of the topological approach to the Mumford conjecture, and [MT] for a more technical discussion.)

2.4. Pointed nodal curves, and the moduli space of stable pointed curves.

As our moduli space $\mathcal{M}_g$ is a smooth orbifold of dimension $3g-3$, it is wonderful in all ways but one: it is not compact. It would be useful to have a good compactification, one that is still smooth, and also has good geometric meaning. This leads us to extend our notion of smooth curves slightly.

A *node* of a curve is a singularity analytically isomorphic to $xy = 0$ in $\mathbb{C}^2$. A *nodal curve* is a curve (compact, connected) smooth away from a finite number of points (possibly zero), which are nodes. An example is sketched in Figure 2, in both "real" and "cartoon" form. One caution with the "real" picture: the two branches at the node are not tangent; this optical illusion arises from the need of our limited brains to represent the picture in three-dimensional space. A *pointed nodal curve* is a nodal curve with the additional data of n distinct smooth points labeled 1 through n (or n distinct labels of your choice, such as p_1 through p_n).

The *geometric genus* of an *irreducible* curve is its genus once all of the nodes are "unglued". For example, the components of the curve in Figure 2 have genus 1 and 0.

We define the *(arithmetic) genus* of a pointed nodal curve informally as the genus of a "smoothing" of the curve, which is indicated in Figure 3. More formally, we define it as $h^1(C, \mathcal{O}_C)$. This notion behaves well with respect to deformations. (More formally, it is locally constant in flat families.)

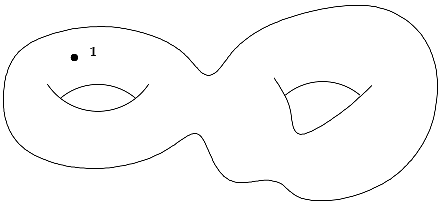

FIGURE 3. By smoothing the curve of Figure 2, we see that its genus is 2

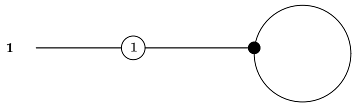

FIGURE 4. The dual graph to the pointed nodal curve of Figure 2 (unlabeled vertices are genus 0)

Exercise (for those with enough background): If C has δ nodes, and its irreducible components have geometric genus $g_1, \dots, g_k$ respectively, show that the arithmetic genus of C is $\sum_{i=1}^{k}(g_i - 1) + 1 + \delta$.

We define the *dual graph* of a pointed nodal curve as follows. It consists of vertices, edges, and "half-edges". The vertices correspond to the irreducible components of the curve, and are labeled with the *geometric* genus of the component. When the genus is 0, the label will be omitted for convenience. The edges correspond to the nodes, and join the corresponding vertices. (Note that an edge can join a vertex to itself.) The half-edges correspond to the labeled points. The dual graph corresponding to Figure 2 is given in Figure 4.

A nodal curve is said to be *stable* if it has finite automorphism group. This is equivalent to a combinatorial condition: (i) each genus 0 vertex of the dual graph has valence at least three, and (iii) each genus 1 vertex has valence at least one.

Exercise. Prove this. You may use the fact that a genus $g \geq 2$ curve has finite automorphism group, and that an elliptic curve (i.e. a 1-pointed genus 1 curve) has finite automorphism group. While you are proving this, you may as well show that the automorphism group of a stable genus 0 curve is trivial.

2.5. *Exercise.* Draw all possible stable dual graphs for $g = 0$ and $n \leq 5$; also for $g = 1$ and $n \leq 2$. In particular, show there are no stable dual graphs if $(g, n) = (0, 0)$, $(0, 1)$, $(0, 2)$, $(1, 0)$.

Fact. There is a moduli space of stable nodal curves of genus g with n marked points, denoted $\overline{\mathcal{M}}_{g,n}$. There is an open subset corresponding to smooth curves, denoted $\mathcal{M}_{g,n}$. The space $\overline{\mathcal{M}}_{g,n}$ is irreducible, of dimension $3g-3+n$, and smooth.

(For Gromov-Witten experts: you can interpret this space as the moduli space of stable maps to a point. But this is in some sense backwards, both historically, and in terms of the importance of both spaces.)

Exercise. Show that $\chi(\mathcal{M}_{g,n}) = (-1)^n \frac{(2g+n-3)! B_{2g}}{2g(2g-2)!}$, using the Harer-Zagier fact earlier (3).

2.6. *Strata.* To each stable graph Γ of genus g with n points, we associate the subset $\mathcal{M}_\Gamma \subset \overline{\mathcal{M}}_{g,n}$ of curves with that dual graph. This translates to the space of curves of a given topological type. Notice that if Γ is the dual graph given in Figure 4, we can obtain any curve in $\mathcal{M}_\Gamma$ by taking a genus 0 curve with three marked points and gluing two of the points together, and gluing the result to a genus 1 curve with two marked points. (This is most clear in Figure 2.) Thus each $\mathcal{M}_\Gamma$ is naturally the quotient of a product of $\mathcal{M}_{g',n'}$'s by a finite group. For example, if Γ is as in Figure 4, $\mathcal{M}_\Gamma = (\mathcal{M}_{0,3} \times \mathcal{M}_{1,2})/S_2$.

These $\mathcal{M}_\Gamma$ give a stratification of $\overline{\mathcal{M}}_{g,n}$, and this stratification is essentially as nice as one could hope. For example, the divisors (the closure of the codimension one strata) meet transversely along smaller strata. The dense open set $\mathcal{M}_{g,n}$ is one stratum; the rest are called *boundary strata*. The codimension 1 strata are called *boundary divisors*.

Notice that even if we were initially interested only in unpointed Riemann surfaces, i.e. in the moduli space $\mathcal{M}_g$, then this compactification forces us to consider $\mathcal{M}_\Gamma$, which in turn forces us to consider pointed nodal curves.

Exercise. By computing $\dim \mathcal{M}_\Gamma$, check that the codimension of the boundary stratum corresponding to a dual graph Γ is precisely the number of edges of the dual graph. (Do this first in some easy case!)

2.7. *Important exercise.* Convince yourself that $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$. The isomorphism is given as follows. Given four distinct points p_1, p_2, p_3, p_4 on a genus 0 curve (isomorphic to $\mathbb{P}^1$), we may take their cross-ratio $\lambda = (p_4 - p_1)(p_2 - p_3)/(p_4 - p_3)(p_2 - p_1)$, and in turn the cross-ratio determines the points $p_1, \dots, p_4$ up to automorphisms of $\mathbb{P}^1$. The cross-ratio can take on any value in $\mathbb{P}^1 - \{0, 1, \infty\}$. The three 0-dimensional strata correspond to these three missing points — figure out which stratum corresponds to which of these three points.

Exercise. Write down the strata of $\overline{\mathcal{M}}_{0,5}$, along with which stratum is in the closure of which other stratum (cf. Exercise 2.5).

2.8. Natural morphisms among these moduli spaces.

We next describe some natural maps between these moduli spaces. For example, given any n-pointed genus g curve (where $(g, n) \neq (0, 3), (1, 1), n > 0$), we can forget the nth point, to obtain an $(n-1)$-pointed nodal curve of genus g. This curve may not be stable, but it can be "stabilized" by contracting all components that are 2-pointed genus 0 curves. This gives us a map $\overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_{g,n-1}$, which we dub the *forgetful morphism*.

Exercise. Create an example of a dual graph where stabilization is necessary. Also, explain why we excluded the cases $(g, n) = (0, 3), (1, 1)$.

2.9. *Important exercise.* Interpret $\overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ as the universal curve over $\overline{\mathcal{M}}_{g,n}$. (This is a bit subtle. Suppose C is a nodal curve, with node p. Which *stable* pointed curve with 1 marked point corresponds to p? Similarly, suppose (C, p) is a pointed curve. Which stable 2-pointed curve corresponds to p?)

Given an $(n_1 + 1)$-pointed curve of genus g_1, and an $(n_2 + 1)$-pointed curve of genus g_2, we can glue the first curve to the second along the last point of each, resulting in an $(n_1 + n_2)$-pointed curve of genus $g_1 + g_2$. This gives a map

$$\overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}.$$

Similarly, we could take a single $(n + 2)$-pointed curve of genus g, and glue its last two points together to get an n-pointed curve of genus $g + 1$; this gives a map

$$\overline{\mathcal{M}}_{g,n+2} \to \overline{\mathcal{M}}_{g+1,n}.$$

We call these last two types of maps *gluing morphisms*.

We call the forgetful and gluing morphisms the *natural morphisms* between moduli spaces of curves.

3. Tautological cohomology classes on moduli spaces of curves, and their structure

We now define some cohomology classes on these two sorts of moduli spaces of curves, $\mathcal{M}_g$ and $\overline{\mathcal{M}}_{g,n}$. Clearly by Harer and Zagier's Euler-characteristic calculation (3), we should expect some interesting classes, and it is a challenge to name some. Inside the cohomology ring, there is a subring, called the *tautological (sub)ring* of the cohomology ring, that consists informally of the geometrically natural classes. An equally informal definition of the tautological ring is: all the classes you can easily think of. (Of course, this isn't a mathematical statement. But we do not know of a single algebraic class in $H^*(\mathcal{M}_g)$ that can be explicitly written down, that is provably not tautological, even though we expect that they exist.) Hence we care very much about this subring.

The reader may work in cohomology, or in the Chow ring (the algebraic analogue of cohomology). The tautological elements will live naturally in either, and the reader can choose what he or she is most comfortable with. In order to emphasize that one can work algebraically, and also that our dimensions and codimensions are algebraic, I will use the notation of the Chow ring A^i, but most readers will prefer to interpret all statements in the cohomology ring. There is a natural map $A^i \to H^{2i}$, and the reader should be conscious of that doubling of the index.

If α is a 0-cycle on a compact orbifold X, then $\int_X \alpha$ is defined to be its degree.

3.1. Tautological classes on $\mathcal{M}_g$, take one.

A good way of producing cohomology classes on $\mathcal{M}_g$ is to take Chern classes of some naturally defined vector bundles.

On the universal curve $\pi : \mathcal{C}_g \to \mathcal{M}_g$ over $\mathcal{M}_g$, there is a natural line bundle $\mathcal{L}$; on the fiber C of $\mathcal{C}_g$, it is the line bundle of differentials of C. Define $\psi := c_1(\mathcal{L})$, which lies in $A^1(\mathcal{C}_g)$ (or $H^2(\mathcal{C}_g)$ — but again, we will stick to the language of A^*). Then $\psi^{i+1} \in A^{i+1}(\mathcal{C}_g)$, and as π is a proper map, we can push this class forward to $\mathcal{M}_g$, to get the *Mumford-Morita-Miller* κ*-class*

$$\boxed{\kappa_i := \pi_* \psi^{i+1}, \quad i = 0, 1, \ldots.}$$

Another natural vector bundle is the following. Each genus g curve (i.e. each point of $\mathcal{M}_g$) has a g-dimensional space of differentials (§2.1), and the corresponding rank g vector bundle on $\mathcal{M}_g$ is called the *Hodge bundle*, denoted $\mathbb{E}$. (It can also be defined by $\mathbb{E} := \pi_* \mathcal{L}$.) We define the λ-classes by

$$\boxed{\lambda_i := c_i(\mathbb{E}), \quad i = 0, \ldots, g.}$$

We define the *tautological ring* as the subring of the Chow ring generated by the κ-classes. (We will have another definition in §3.8.) This ring is denoted $R^*(\mathcal{M}_g) \subset A^*(\mathcal{M}_g)$ (or $R^*(\mathcal{M}_g) \subset H^{2*}(\mathcal{M}_g)$).

It is a miraculous "fact" that everything else you can think of seems to lie in this subring. For example, the following generating function identity determines the λ-classes from the κ-classes in an attractive way, and incidentally serves as an advertisement for the fact that generating functions (with coefficients in the Chow ring) are a good way to package information [Fab1, p. 111]:

$$\sum_{i=0}^{\infty} \lambda_i t^i = \exp\left(\sum_{i=1}^{\infty} \frac{B_{2i} \kappa_{2i-1}}{2i(2i-1)} t^{2i-1} \right).$$

3.2. Faber's conjectures.

The study of the tautological ring was begun in Mumford's fundamental paper [Mu], but there was no reason to think that it was particularly well-behaved. But just over a decade ago, Carel Faber proposed a remarkable constellation of conjectures (first in print in [Fab1]), suggesting that the tautological ring has a beautiful combinatorial structure. It is reasonable to state that Faber's conjectures have motivated a great deal of the remarkable progress in understanding the topology of the moduli space of curves over the last decade.

Although Faber's conjectures deal just with the moduli of smooth curves, their creation required knowledge of the compactification, and even of Gromov-Witten theory, as we will later see.

A good portion of Faber's conjectures can be informally summarized as: "$R^*(\mathcal{M}_g)$ behaves like the ((p,p)-part of the) cohomology ring of a $(g-2)$-dimensional complex projective manifold." We now describe (most of) Faber's conjectures more precisely. I have chosen to cut them into three pieces.

I. "Vanishing/socle" conjecture. $R^i(\mathcal{M}_g) = 0$ for $i > g-2$, and $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$. This was proved by Looijenga [Lo] and Faber [Fab1, Thm. 2]. (Looijenga's theorem will be stated explicitly below, see Theorem 4.5.) We will prove the "vanishing" part $R^i(\mathcal{M}_g) = 0$ for $i > g-2$ in §4.4, and show that $R^{g-2}(\mathcal{M}_g)$ is generated by a single element as a consequence of Theorem 7.10. These statements comprise Looijenga's theorem (Theorem 4.5). The remaining part (that this generator $R^{g-2}(\mathcal{M}_g)$ is non-zero) is a theorem of Faber's, and we omit its proof.

II. Perfect pairing conjecture. The analog of Poincaré duality holds: for $0 \leq i \leq g-2$, the natural product $R^i(\mathcal{M}_g) \times R^{g-2-i}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$ is a perfect pairing. This conjecture is currently completely open, and is only known in special cases.

We call a ring satisfying **I** and **II** a *Poincaré duality ring of dimension* $g-2$.

A little thought will convince you that, thanks to **II**, if we knew the "top intersections" (i.e. the products of κ-classes of total degree $g-2$, as a multiple of the generator of $R^{g-2}(\mathcal{M}_g)$), then we would know the complete structure of the tautological ring. Faber predicts the answer to this as well.

III. Intersection number conjecture (take one). (We will give a better statement in Conjecture 3.23, in terms of a partial compactification of $\mathcal{M}_{g,n}$.) For any n-tuple of non-negative integers $(d_1, \ldots, d_n)$ summing to $g-2$,

$$\frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod_{j=1}^n (2d_j+1)!!}\kappa_{g-2} = \sum_{\sigma \in S_n} \kappa_\sigma \tag{4}$$

where if $\sigma = (a_{1,1} \cdots a_{1,i_1})(a_{2,1} \cdots a_{2,i_2}) \cdots$ is the cycle decomposition of σ, then κ_σ is defined to be $\prod_j \kappa_{d_{a_{j,1}} + d_{a_{j,2}} + \cdots + d_{a_{j,i_j}}}$. Recall that $(2k-1)!! = 1 \times 3 \times \cdots \times (2k-1) = (2k)!/2^k k!$.

For example, we have

$$\kappa_{i-1}\kappa_{g-i-1} + \kappa_{g-2} = \frac{(2g-1)!!}{(2i-1)!!(2g-2i-1)!!}\kappa_{g-2}$$

and

$$\kappa_1^{g-2} = \frac{1}{g-1} 2^{2g-5}(g-2)!^2 \kappa_{g-2}.$$

Remarkably, Faber was able to deduce this elegant conjecture from a very limited amount of experimental data.

Faber's intersection number conjecture begs an obvious question: why is this formula so combinatorial? What is the combinatorial structure behind this ring? Faber's alternate description of the intersection number conjecture (Conjecture 3.23) will be even more patently combinatorial.

Faber's intersection number conjecture is now a theorem. Getzler and Pandharipande showed that it is a formal consequence of the *Virasoro conjecture* for the projective plane [GeP]. The Virasoro conjecture is due to the physicists Eguchi, Hori, Xiong, and also the mathematician Sheldon Katz, and deals with the Gromov-Witten invariants of some space X. (See [CK, Sect. 10.1.4] for a statement.) Getzler and Pandharipande show that the Virasoro conjecture in $\mathbb{P}^2$ implies a recursion among the intersection numbers on the (compact) moduli space of stable curves, which in turn is equivalent to a recursion for the top intersections in Faber's conjecture. They then show that the recursions have a unique solution, and that Faber's prediction is a solution.

Givental has announced a proof of the Virasoro conjecture for projective space (and more generally Fano toric varieties) [Giv]. The details of the proof have not appeared, but Y.-P. Lee and Pandharipande are writing a book [LeeP] giving the details. This theorem is really a tour-de-force, and the most important result in Gromov-Witten theory in some time. However, it seems a roundabout and high-powered way of proving Faber's intersection number conjecture. For example, by its nature, it cannot shed light on the combinatorial structure behind the intersection numbers. For this reason, it seems worthwhile giving a more direct argument. At the end of these notes, I will outline a program for tackling this conjecture (joint with the combinatorialists I.P. Goulden and D.M. Jackson), and a proof in a large class of cases.

(There are two other conjectures in this constellation worth mentioning. Faber conjectures that $\kappa_1, \dots, \kappa_{[g/3]}$ generate the tautological ring, with no relations in degrees $\leq [g/3]$. Both Morita [Mo1] and Ionel [I2] have given proofs of the first part of this conjecture a few years ago. Faber also conjectures that $R^*(\mathcal{M}_g)$ satisfies the Hard Lefschetz and Hodge Positivity properties with respect to the class κ_1 [Fab1, Conj. 1(bis)].

As evidence, Faber has checked that his conjectures hold true in genus up to 21 [Fab4]. I should emphasize that this check is very difficult to do — the rings in question are quite large and complicated! Faber's verification involves some clever constructions, and computer-aided computations.

Morita has recently announced a conjectural form of the tautological ring, based on the representation theory of the symplectic group $\mathrm{Sp}(2g, \mathbb{Q})$ [Mo2, Conj. 1]. This is a new and explicit (and attractive) proposed description of the tautological ring. One might hope that his conjecture may imply Faber's conjecture, and may also be provable.

3.3. Tautological classes on $\overline{\mathcal{M}}_{g,n}$.

We can similarly define a tautological ring on the compact moduli space of stable pointed curves, $\overline{\mathcal{M}}_{g,n}$. In fact here the definition is cleaner, and even sheds new light on the tautological ring of $\mathcal{M}_g$. As before, this ring includes "all classes one can easily think of", and as before, it will be most cleanly described in terms of Chern classes of natural vector bundles. Before we give a formal definition, we begin by discussing some natural classes on $\overline{\mathcal{M}}_{g,n}$.

3.4. *Strata.* We note first that we have some obvious (co)homology classes on $\overline{\mathcal{M}}_{g,n}$, that we didn't have on $\mathcal{M}_g$: the fundamental classes of the (closure of the) strata. We will discuss these classes and their relations at some length before moving on.

In genus 0 (i.e., on $\overline{\mathcal{M}}_{0,n}$), the cohomology (and Chow) ring is generated by these classes. (The reason is that each stratum of the boundary stratification is by (Zariski-)open subsets of affine space.) We will see why the tautological groups are generated by strata in Exercise 4.9.

We thus have generators of the cohomology groups; it remains to find the relations. On $\overline{\mathcal{M}}_{0,4}$, the situation is especially nice. We have checked that $\overline{\mathcal{M}}_{0,4}$ is isomorphic to $\mathbb{P}^1$ (Exercise 2.7), and there are three boundary points. They are homotopic (as any two points on $\mathbb{P}^1$ are homotopic) — and even *rationally equivalent*, the algebraic version of homotopic in the theory of Chow groups.

By pulling back these relations by forgetful morphisms, and pushing forward by gluing morphisms, we get many other relations for various $\overline{\mathcal{M}}_{0,n}$. We dub these *cross-ratio relations*, although they go by many other names in the literature. Keel has shown that these are *all* the relations [Ke].

In genus 1, the tautological ring (although not the cohomology or Chow rings!) are again generated by strata. (We will see why in Exercise 3.28, and again in Exercise 4.9.) We again have cross-ratio relations, induced by a single (algebraic/complex) codimension 1 relation on $\overline{\mathcal{M}}_{0,4}$. Getzler proved a new (codimension 2) relation on $\overline{\mathcal{M}}_{1,4}$ [Ge1, Thm. 1.8] (now known as *Getzler's relation*). (It is remarkable that this relation, on an important compact smooth fourfold, parametrizing four points on elliptic curves, was discovered so late.) Via the natural morphisms, this induces relations on $\overline{\mathcal{M}}_{1,n}$ for all n. Some time ago, Getzler announced that these two sorts of relations were the only relations among the strata [Ge1, par. 2].

In genus 2, there are very natural cohomology classes that are not combinations of strata, so it is now time to describe other tautological classes.

3.5. *Other tautological classes.* Once again, we can define classes as Chern classes of natural vector bundles.

On $\overline{\mathcal{M}}_{g,n}$, for $1 \leq i \leq n$, we define the line bundle $\mathbb{L}_i$ as follows. On the universal curve $\mathcal{C}_{g,n} \to \overline{\mathcal{M}}_{g,n}$, the cotangent space at the fiber above $[(C, p_1, \dots, p_n)] \in \overline{\mathcal{M}}_{g,n}$ at point p_i is a one-dimensional vector space, and this vector space varies smoothly with $[(C, p_1, \dots, p_n)]$. This is $\mathbb{L}_i$. More precisely, if $s_i : \overline{\mathcal{M}}_{g,n} \to \mathcal{C}_{g,n}$ is the section of π corresponding to the ith marked point, then $\mathcal{L}_i$ is the pullback by s_i of the sheaf of relative differentials or the relative dualizing sheaf (it doesn't matter which, as the section meets only the smooth locus). Define $\psi_i = c_1(\mathbb{L}_i) \in A^1(\overline{\mathcal{M}}_{g,n})$.

A genus g nodal curve has a g-dimensional vector space of sections of the dualizing line bundle. These vector spaces vary smoothly, yielding the *Hodge bundle* $\mathbb{E}_{g,n}$ on $\overline{\mathcal{M}}_{g,n}$. (More precisely, if π is the universal curve over $\overline{\mathcal{M}}_{g,n}$, and $\mathcal{K}_\pi$ is the relative dualizing line bundle on the universal curve, then $\mathbb{E}_{g,n} := \pi_* \mathcal{K}_\pi$.) Define $\lambda_i := c_i(\mathbb{E}_{g,n})$ on $\overline{\mathcal{M}}_{g,n}$. Clearly the restriction of the Hodge bundle and λ-classes from $\overline{\mathcal{M}}_g$ to $\mathcal{M}_g$ are the same notions defined earlier.

Similarly, there is a more general definition of κ-classes, due to Arbarello and Cornalba [ArbC].

One might reasonably hope that these notions should behave well under the forgetful morphism $\pi : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$ (which we can interpret as the universal curve by Exercise 2.9).

Exercise. Show that there is a natural isomorphism $\pi^* \mathbb{E}_{g,n} \cong \mathbb{E}_{g,n+1}$, and hence that $\pi^* \lambda_k = \lambda_k$. (Caution: the two λ_k's in this statement are classes on two different spaces.)

The behavior of the ψ-classes under pullback by the forgetful morphism has a slight twist.

3.6. *Comparison lemma.* — $\psi_1 = \pi^* \psi_1 + D_{0,\{1,n+1\}}$.

(Caution: the two ψ_1's in the comparison lemma are classes on two different spaces!) Here $D_{0,\{1,n+1\}}$ means the boundary divisor corresponding to reducible curves with one node, where one component is genus 0 and contains only the marked points p_1 and p_{n+1}. The analogous statement applies with 1 replaced by any number up to n of course.

Exercise (for people with more background). Prove the Comparison lemma 3.6. (Hint: First show that we have equality away from $D_{0,\{1,n+1\}}$. Hence $\psi_1 = \pi^* \psi_1 + k D_{0,\{1,n+1\}}$ for some integer k, and this integer k can be computed on a single test family.)

As an application:

3.7. *Exercise.* Show that ψ_1 on $\overline{\mathcal{M}}_{0,4}$ is $\mathcal{O}(1)$ (where $\overline{\mathcal{M}}_{0,4} \cong \mathbb{P}^1$, Exercise 2.7).

Exercise. Express ψ_1 explicitly as a sum of boundary divisors on $\overline{\mathcal{M}}_{0,n}$.

We are now ready to define the tautological ring of $\overline{\mathcal{M}}_{g,n}$. We do this by defining the rings for all g and n at once.

3.8. *Definition.* The system of tautological rings $(R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}))_{g,n}$ (as g and n vary over all genera and numbers of marked points) is the smallest system of $\mathbb{Q}$-algebras closed under pushforwards by the natural morphisms.

This elegant definition is due to Faber and Pandharipande [FabP3, §0.1].

Define the tautological ring of any open subset of $\overline{\mathcal{M}}_{g,n}$ by its restriction from $\overline{\mathcal{M}}_{g,n}$. In particular, we can recover our original definition of the tautological ring of $\mathcal{M}_g$ (§3.1).

It is a surprising fact that everything else you can think of (such as ψ-classes, λ-classes and κ-classes) will lie in this ring. (It is immediate that fundamental classes of strata lie in this ring: they are pushforwards of the fundamental classes of their "component spaces", cf. §2.6.)

We next give an equivalent description of the tautological *groups*, which will be convenient for many of our arguments, because we do not need to make use of the multiplicative structure. In this description, the ψ-classes play a central role.

3.9. *Definition* [GrV3, Defn. 4.2]. The system of tautological rings $(R^*(\overline{\mathcal{M}}_{g,n}) \subset A^*(\overline{\mathcal{M}}_{g,n}))_{g,n}$ is the smallest system of $\mathbb{Q}$-vector spaces closed under pushforwards by the natural morphisms, such that all monomials in $\psi_1, \dots, \psi_n$ lie in $R^*(\overline{\mathcal{M}}_{g,n})$.

The equivalence of Definition 3.8 and Definition 3.9 is not difficult (see for example [GrV3]).

3.10. Faber-type conjectures for $\overline{\mathcal{M}}_{g,n}$, and the conjecture of Hain-Looijenga-Faber-Pandharipande.

In analogy with Faber's conjecture, we have the following.

3.11. Conjecture. $R^*(\overline{\mathcal{M}}_{g,n})$ is a Poincaré-duality ring of dimension $3g-3+n$.

This was first asked as a question by Hain and Looijenga [HLo, Question 5.5], first stated as a speculation by Faber and Pandharipande [FabP1, Speculation 3] (in the case $n = 0$), and first stated as a conjecture by Pandharipande [P, Conjecture 1]. In analogy with Faber's conjecture, we break this into two parts.

I. "Socle" conjecture. $R^{3g-3+n}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$. This is obvious if we define the tautological ring in terms of cohomology: $H^{2(3g-3+n)}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$, and the zero-dimensional strata show that the tautological zero-cycles are not all zero. However, in the tautological Chow ring, the socle conjecture is not at all obvious. Moreover, the conjecture is not true in the full Chow ring — $A_0(\overline{\mathcal{M}}_{1,11})$ is uncountably generated, while the conjecture states that $R_0(\overline{\mathcal{M}}_{1,11})$ has a single generator. (By R_0, we of course mean R^{3g-3+n}.)

We will prove the vanishing conjecture in §4.6.

II. Perfect pairing conjecture For $0 \leq i \leq 3g-3+n$, the natural product

$$R^i(\overline{\mathcal{M}}_{g,n}) \times R^{3g-3+n-i}(\overline{\mathcal{M}}_{g,n}) \to R^{3g-3+n}(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$$

is a perfect pairing. (We currently have no idea why this should be true.)

Hence, in analogy with Faber's conjecture, if this conjecture were true, then we could recover the entire ring by knowing the top intersections. This begs the question of how to compute all top intersections.

3.12. Fact/recipe (Mumford and Faber). If we knew the top intersections of ψ-classes, we would know all top intersections. In other words, there is an algorithm to compute all top intersections if we knew the numbers

$$\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n}, \qquad \sum a_i = 3g - 3 + n. \tag{5}$$

(This is a worthwhile exercise for people with some familiarity with the moduli space of curves.) This is the basis of Faber's wonderful computer program [Fab2] computing top intersections of various tautological classes. For more information, see [Fab3]. This construction is useful in understanding the definition (Defn. 3.9) of the tautological group in terms of the ψ-classes.

Until a key insight of Witten's, there was no a priori reason to expect that these numbers should behave nicely. We will survey three methods of computing these numbers: (i) partial results in low genus; (ii) Witten's conjecture; and (iii) via the ELSV formula. A fourth (attractive) method was given in Kevin Costello's thesis [C].

3.13. *Top intersections on $\overline{\mathcal{M}}_{g,n}$: partial results in low genus.* Here are two crucial relations among top intersections.

Dilaton equation. If $\overline{\mathcal{M}}_{g,n}$ exists (i.e. there are stable n-pointed genus g curves, or equivalently $2g-2+n>0$), then

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\beta_1}\psi_2^{\beta_2}\cdots\psi_n^{\beta_n}\psi_{n+1} = (2g-2+n)\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\beta_1}\cdots\psi_n^{\beta_n}.$$

String equation. If $2g-2+n>0$, then

$$\int_{\overline{\mathcal{M}}_{g,n+1}} \psi_1^{\beta_1}\psi_2^{\beta_2}\cdots\psi_n^{\beta_n} = \sum_{i=1}^{n}\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{\beta_1}\psi_2^{\beta_2}\cdots\psi_i^{\beta_i-1}\cdots\psi_n^{\beta_n}$$

(where you ignore terms where you see negative exponents).

Exercise (for those with more experience). Prove these using the Comparison lemma 3.6.

Equipped with the string equation alone, we can compute all top intersections in genus 0, i.e. $\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{\beta_1}\cdots\psi_n^{\beta_n}$ where $\sum \beta_i = n-3$. (In any such expression, some β_i must be 0, so the string equation may be used.) Thus we can recursively solve for these numbers, starting from the base case $\int_{\overline{\mathcal{M}}_{0,3}} \psi_1^0\psi_2^0\psi_3^0 = 1$.

Exercise. Show that

$$\int_{\overline{\mathcal{M}}_{0,n}} \psi_1^{a_1}\cdots\psi_n^{a_n} = \binom{n-3}{a_1,\cdots,a_n}.$$

In genus 1, the story is similar. In this case, we need both the string and dilaton equation.

Exercise. Show that any integral

$$\int_{\overline{\mathcal{M}}_{1,n}} \psi_1^{\beta_1} \cdots \psi_n^{\beta_n}$$

can be computed using the string and dilaton equation from the base case $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = 1/24$.

We now sketch why the base case $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = 1/24$ is true. We calculate this by choosing a finite cover $\mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$. Consider a general pencil of cubics in the projective plane. In other words, take two general homogeneous cubic polynomials f and g in three variables, and consider the linear combinations of f and g. The non-zero linear combinations modulo scalars are parametrized by a $\mathbb{P}^1$. Thus we get a family of cubics parametrized by $\mathbb{P}^1$, i.e. $\mathcal{C} \to \mathbb{P}^1$.

You can verify that in this family, there will be twelve singular fibers, that are cubics with one node. One way of verifying this is as follows: $f = g = 0$ consists of nine points $p_1, \ldots, p_9$ (basically by Bezout's theorem — you expect two cubics to meet at nine points). There is a map $\mathbb{P}^2 - \{p_1, \ldots, p_9\} \to \mathbb{P}^1$. If $\mathcal{C}$ is the blow-up of $\mathbb{P}^2$ at the nine points, then this map extends to $\mathcal{C} \to \mathbb{P}^1$, and this is the total space of the family. The (topological) Euler characteristic of $\mathcal{C}$ is the Euler characteristic of $\mathbb{P}^2$ (which is 3) plus 9 (as each blow-up replaces a point by a $\mathbb{P}^1$), i.e. $\chi(\mathcal{C}) = 12$. Considering $\mathcal{C}$ as a fibration over $\mathbb{P}^1$, most fibers are elliptic curves, which have Euler characteristic 0. Hence $\chi(\mathcal{C})$ is the sum of the Euler characteristics of the singular fibers. Each singular fiber is a nodal cubic, which is isomorphic to $\mathbb{P}^1$ with two points glued together (depicted in Figure 5); this is the union of $\mathbb{C}^*$ (which has Euler characteristic 0) with a point, so $\chi(\mathcal{C})$ is the number of singular fibers. (This argument needs further justification at every point!)

We have a section of $\mathcal{C} \to \mathbb{P}^1$, given by the exceptional fiber E of the blow-up of p_1. Hence we have a moduli map $\mu : \mathbb{P}^1 \to \overline{\mathcal{M}}_{1,1}$ of smooth curves. Clearly it doesn't map $\mathbb{P}^1$ to a point, as some of the fibers are smooth, and twelve are singular. Thus the moduli map μ is surjective (as the image is an irreducible closed set that is not a point). You might suspect that μ has degree 12, as the preimage of the boundary divisor $\Delta \in \overline{\mathcal{M}}_{1,1}$ has 12 preimages, and one can check that μ is nonsingular here. However, we come to one of the twists of stack theory — each point of $\overline{\mathcal{M}}_{1,1}$, including Δ, has degree $1/2$ — each point should be counted with multiplicity one over the size of its automorphism group, and each 1-pointed genus 1 stable curve has precisely one nontrivial automorphism.

Thus $24 \int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = \int_{\mathbb{P}^1} \mu^* \psi_1$, so we wish to show that $\int_{\mathbb{P}^1} \mu^* \psi_1 = 1$. This is an explicit computation on $\mathcal{C} \to \mathbb{P}^1$. You may check that on the blow-up to $\mathcal{C}$, the dualizing sheaf to the fiber at p_1 is given by $-\mathcal{O}(E)|_E$. As $E^2 = -1$, we have $\int_{\mathbb{P}^1} \mu^* \psi_1 = -E^2 = 1$ as desired.

In higher genus, the string and dilaton equation are also very useful.

Exercise. Fix g. Show that using the string and dilaton equation, all of the numbers (5) (for all n) can be computed from a finite number of base cases. The number of base cases required is the number of partitions of $3g-3$. (It is useful to describe this more precisely, by explicitly describing the generating function for (5) in terms of these base cases.)

3.14. *Witten's conjecture.* So how do we get at these remaining base cases? The answer was given by Witten [W]. (This presentation is not chronological — Witten's conjecture came first, and motivated most of what followed. In particular, it predates Faber's conjectures, and was used to generate the data that led Faber to his conjectures.)

Witten's conjecture (Kontsevich's theorem). Let

$$F_g = \sum_{n \geq 0} \frac{1}{n!} \sum_{k_1, \dots, k_n} \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{k_1} \cdots \psi_n^{k_n} \right) t_{k_1} \cdots t_{k_n}$$

be the generating function for the genus g numbers (5), and and let

$$F = \sum F_g \hbar^{2g-2}$$

be the generating function for all genera. (This is *Witten's free energy*, or the *Gromov-Witten potential of a point*.) Then

$$(2n+1)\frac{\partial^3}{\partial t_n \partial t_0^2} F = \left(\frac{\partial^2}{\partial t_{n-1} \partial t_0} F \right) \left(\frac{\partial^3}{\partial t_0^3} F \right) + 2 \left(\frac{\partial^3}{\partial t_{n-1} \partial t_0^2} F \right) \left(\frac{\partial^2}{\partial t_0^2} F \right) + \frac{1}{4} \frac{\partial^5}{\partial t_{n-1} \partial t_0^4} F.$$

Witten's conjecture now has many proofs, by Kontsevich [Ko1], Okounkov-Pandharipande [OP], Mirzakhani [Mi], and Kim-Liu [KiL]. It is a sign of the richness of this conjecture that these proofs are all very different, and all very enlightening in different ways.

The reader should not worry about the details of this formula, and should just look at its shape. Those familiar with integrable systems will recognize this as the Korteweg-de Vries (KdV) equation, in some guise. There was a later reformulation due to Dijkgraaf, Verlinde, and Verlinde [DVV], in terms of the Virasoro algebra. Once again, the reader should not worry about the precise statement, and concentrate on the form of the conjecture. Define differential operators ($n \geq -1$)

$$\begin{aligned}
L_{-1} &= -\frac{\partial}{\partial t_0} + \frac{\hbar^{-2}}{2} t_0^2 + \sum_{i=0}^{\infty} t_{i+1} \frac{\partial}{\partial t_i} \\
L_0 &= -\frac{3}{2} \frac{\partial}{\partial t_1} + \sum_{i=0}^{\infty} \frac{2i+1}{2} t_i \frac{\partial}{\partial t_i} + \frac{1}{16} \\
L_n &= \sum_{k=0}^{\infty} \frac{\Gamma(m+n+\frac{3}{2})}{\Gamma(k+\frac{1}{2})} (t_k - \delta_{k,1}) \frac{\partial}{\partial t_{n+k}} + \frac{\hbar^2}{2} \sum_{k=1}^{n-1} (-1)^{k+1} \frac{\Gamma(n-k+\frac{1}{2})}{\Gamma(-k-\frac{1}{2})} \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_{n-k-1}} \qquad (n > 0)
\end{aligned}$$

These operators satisfy $[L_m, L_n] = (m-n) L_{m+n}$.

Exercise. Show that $L_{-1} e^F = 0$ is equivalent to the string equation. Show that $L_0 e^F = 0$ is equivalent to the dilaton equation.

Witten's conjecture is equivalent to $L_n e^F = 0$ for all n. These equations let you inductively solve for the coefficients of F, and hence compute all these numbers.

3.15. *The Virasoro conjecture.* The Virasoro formulation of Witten's conjecture has a far-reaching generalization, the *Virasoro conjecture* described earlier. Instead of top intersections on the moduli space of curves, it addresses top (virtual) intersections on the moduli space of maps of curves to some space X. Givental's proof (to be explicated by Lee and Pandharipande) for the case of projective space (and more generally Fano toric varieties) was mentioned earlier. It is also worth mentioning Okounkov and Pandharipande's proof in the case where X is a curve; this is also a tour-de-force.

3.16. *Hurwitz numbers and the ELSV formula.* We can also recover these top intersections via the old-fashioned theme of branched covers of the projective line, the very technique that let us compute the dimension of the moduli space of curves, and of the Picard variety §2.2.

Fix a genus g, a degree d, and a partition of d into n parts, $\alpha_1 + \cdots + \alpha_n = d$, which we write as $\alpha \vdash d$. Let

$$r := 2g + d + n - 2. \tag{6}$$

Fix $r+1$ points $p_1, \ldots, p_r, \infty \in \mathbb{P}^1$. Define the *Hurwitz number* H^g_α to be the number of branched covers of $\mathbb{P}^1$ by a Riemann surface, that are unbranched away from $p_1, \ldots, p_r, \infty$, such that the branching over ∞ is given by $\alpha_1, \ldots, \alpha_n$ (i.e. there are n preimages of ∞, and the branching at the ith preimage is of order α_i, i.e. the map is analytically locally given by $t \mapsto t^{\alpha_i}$), and there is the simplest possible branching over each p_i, i.e. the branching is given by $2 + 1 + \cdots + 1 = d$. (To describe this *simple branching* more explicitly: above any such branch point, $d-2$ of the sheets are unbranched, and the remaining two sheets come together. The analytic picture of the two sheets is the projection of the parabola $y^2 = x$ to the x-axis in $\mathbb{C}^2$.) We consider the n preimages of ∞ to be labeled. Caution: in the literature, sometimes the preimages of ∞ are *not labeled*; that definition of Hurwitz number will be smaller than ours by a factor of $\# \operatorname{Aut} \alpha$, where $\operatorname{Aut} \alpha$ is the subgroup of S_n fixing the n-tuple $(\alpha_1, \ldots, \alpha_n)$ (e.g. if $\alpha = (2,2,2,5,5)$, then $\# \operatorname{Aut} \alpha = 3!2!$).

One technical point: each cover is counted with multiplicity 1 over the size of the automorphism group of the cover.

Exercise. Use the Riemann-Hurwitz formula (2) to show that if the cover is connected, then it has genus g.

Experts will recognize these as relative descendant Gromov-Witten invariants of $\mathbb{P}^1$; we will discuss relative Gromov-Witten invariants of $\mathbb{P}^1$ in Section 5. However, they are something much more down-to-earth. The following result shows that this number is a purely combinatorial object. In particular, there are a finite number of such covers.

3.17. *Proposition.* —

$$H^g_\alpha = \# \left\{ (\sigma_1, \ldots, \sigma_r) : \sigma_i \textit{ transpositions generating } S_d, \prod_{i=1}^r \sigma_i \in C(\alpha) \right\} \# \operatorname{Aut} \alpha / d!,$$

where the σ_i are transpositions generating the symmetric group S_d, and $\mathcal{C}(\alpha)$ is the conjugacy class in S_d corresponding to partition α.

Before we give the proof, we make some preliminary comments. As an example, consider $d = 2$, $\alpha = 2$, g arbitrary, so $r = 2g + 1$. The above formula gives $H^g_\alpha = 1/2$, which at first blush seems like nonsense — how can we count covers and get a non-integer? Remember however the combinatorial/stack-theoretic principal that objects should be counted with multiplicity 1 over the size of their automorphism group. Any double cover of this sort always has a non-trivial involution (the "hyperelliptic involution"). Hence there is indeed one cover, but it is counted as "half a cover". Fortunately, this is the only case of Hurwitz numbers for which this is an issue. The reader may want to follow this particular case through in the proof.

Proof of Proposition 3.17. Pick another point $0 \in \mathbb{P}^1$ distinct from $p_1, \dots, p_r, \infty$. Choose branch cuts from 0 to $p_1, \dots, p_r, \infty$ (non-intersecting paths from 0 to p_1, 0 to p_2, …, 0 to ∞) such that their cyclic order around 0 is $p_1, \dots, p_r, \infty$. Suppose $C \to \mathbb{P}^1$ is one of the branched covers counted by H^g_α. Then label the d preimages of 0 with 1 through d in some way. We will count these labeled covers, and divide by $d!$ at the end. Now cut along the preimages of the branch-cuts. As $\mathbb{P}^1$ minus the branch-cuts is homeomorphic to a disc, which is simply connected, its preimage must be d copies of the disc, labelled 1 through d according to the label on the preimage of 0. We may reconstruct $C \to \mathbb{P}^1$ by determining how to glue these sheets together along the branch cuts. The monodromy of the cover $C \to \mathbb{P}^1$ around p_i is an element σ_i of S_d, and this element will be a transposition, corresponding to the two sheets being interchanged above that branch point. Similarly, the monodromy around ∞ is also an element σ_∞ of S_d, with cycle type α. The cover has the additional data of the bijection of the cycles with the parts of α. In $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_r, \infty\})$, the loops around p_1, …, p_r, ∞ multiply to the identity, so $\sigma_1\sigma_2 \cdots \sigma_r\sigma_\infty = e$. (Here we use the fact that the branch cuts meet 0 in this particular order.) Thus $\sigma_\infty^{-1} = \sigma_1 \cdots \sigma_r$. This is the only relation among these generators of $\pi_1(\mathbb{P}^1 - \{p_1, \dots, p_r, \infty\}$. Furthermore, the cover C is connected, meaning that we can travel from any one of the d sheets to any of the others, necessarily by travelling around the branch points. This implies that the σ_1, …, σ_r, σ_∞ (and hence just the σ_1, …, σ_r) generate a transitive subgroup of S_d. But the only transitive subgroup of S_d containing a transposition σ_1 is all of S_d.

Conversely, given the data of transposition σ_1, …, σ_r generating S_d, with product of cycle type α, along with a labelling of the parts of the product (of which there are $\# \operatorname{Aut} \alpha$), we can construct a connected cover $C \to \mathbb{P}^1$, by the Riemann existence theorem. Thus, upon forgetting the labels of the d sheets, we obtain the desired equality. $\square$

The above proof clearly extends to deal with more general Hurwitz numbers, where arbitrary branching is specified over each of a number of points.

Proposition 3.17 shows that any Hurwitz number may be readily computed by hand or by computer. What is interesting is the structure behind them. In 1891, Hurwitz [H] showed that

$$H^0_\alpha = r! d^{n-3} \prod \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right). \tag{7}$$

By modern standards, he provided an outline of a proof. His work was forgotten by a large portion of the mathematics community, and later people proved special cases, including Dénes [D] in the case $n = 1$, Arnol'd [Arn] in the case $n = 2$. In the case $n = d$ (so $\alpha = 1^d$) it was stated by the physicists Crescimanno and Taylor [CT], who apparently asked the combinatorialist Richard Stanley about it, who in turn asked Goulden and Jackson. Goulden and Jackson independently discovered and proved Hurwitz' original theorem in the mid-nineties [GJ1]. Since then, many proofs have been given, including one by myself using moduli of curves [V1].

Goulden and Jackson studied the problem for higher genus, and conjectured a structural formula for Hurwitz numbers in general. Their polynomiality conjecture [GJ2, Conj. 1.2] implies the following.

3.18. *Goulden-Jackson Polynomiality Conjecture (one version). — For each g, n, there is a symmetric polynomial $P_{g,n}$ in n variables, with monomials of homogeneous degree between $2g - 3 + n$ and $3g - 3 + n$, such that*

$$H^g_\alpha = r! \prod_{i=1}^{n} \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) P_{g,n}(\alpha_1, \dots, \alpha_n).$$

The reason this conjecture (and the original version) is true is an amazing theorem of Ekedahl, Lando, M. Shapiro, and Vainshtein.

3.19. *Theorem (ELSV formula, by Ekedahl, Lando, M. Shapiro, and Vainshtein* [ELSV1, ELSV2]*). —*

$$H^g_\alpha = r! \prod_{i=1}^{n} \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) \int_{\overline{\mathcal{M}}_{g,n}} \frac{1 - \lambda_1 + \cdots + (-1)^g \lambda_g}{(1 - \alpha_1 \psi_1) \cdots (1 - \alpha_n \psi_n)} \tag{8}$$

(if $\overline{\mathcal{M}}_{g,n}$ exists).

We will give a proof in §6.1.

Here is how to interpret the right side of the equation. Note that the α_i are integers, and the ψ_i's and λ_k's are cohomology (or Chow) classes. Formally invert the denominator, e.g.

$$\frac{1}{1 - \alpha_1 \psi_1} = 1 + \alpha_1 \psi_1 + \alpha_1^2 \psi_1^2 + \cdots .$$

Then multiply everything out inside the integral sign, and discard all but the summands of total codimension $3g-3+n$ (i.e. dimension 0). Then take the degree of this cohomology class.

For example, if $g = 0$ and $n = 4$, we get

$$\begin{aligned}
H^g_\alpha &= r! \prod_{i=1}^4 \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) \int_{\overline{\mathcal{M}}_{0,4}} \frac{1 - \lambda_1 + \cdots \pm \lambda_g}{(1 - \alpha_1 \psi_1) \cdots (1 - \alpha_4 \psi_4)} \\
&= r! \prod_{i=1}^4 \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) \int_{\overline{\mathcal{M}}_{0,4}} (1 + \alpha_1 \psi_1 + \cdots) \cdots (1 + \alpha_4 \psi_4 + \cdots) \\
&= r! \prod_{i=1}^4 \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) \int_{\overline{\mathcal{M}}_{0,4}} (\alpha_1 \psi_1 + \cdots + \alpha_4 \psi_4) \\
&= r! \prod_{i=1}^4 \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) (\alpha_1 + \cdots + \alpha_4) \qquad \text{(Exercise 3.7)} \\
&= r! \prod_{i=1}^4 \left(\frac{\alpha_i^{\alpha_i}}{\alpha_i!} \right) d.
\end{aligned}$$

Exercise. Recover Hurwitz' original formula (7) from the ELSV-formula, at least if $n \geq 3$.

More generally, expanding the integrand of (8) yields

$$\sum_{a_1 + \cdots + a_n + k = 3g-3+n} \left((-1)^k \left(\int_{\overline{\mathcal{M}}_{g,n}} \psi_1^{a_1} \cdots \psi_n^{a_n} \lambda_k \right) \right) (\alpha_1^{a_1} \cdots \alpha_n^{a_n}). \tag{9}$$

This is a polynomial in $\alpha_1, \ldots, \alpha_n$ of homogeneous degree between $2g - 3 + n$ and $3g - 3 + n$. Thus this explains the mystery polynomial in the Goulden-Jackson Polynomiality Conjecture 3.18 — and the coefficients turn out to be top intersections on the moduli space of curves! (The original polynomiality conjecture was actually different, and some translation is necessary in order to make the connection with the ELSV formula [GJV1].)

There are many other consequences of the ELSV formula; see [ELSV2, GJV1] for surveys.

We should take a step back to see how remarkable the ELSV formula is. To any reasonable mathematician, Hurwitz numbers (as defined by Proposition 3.17) are purely discrete, combinatorial objects. Yet their structure is fundamentally determined by the topology of the moduli space of curves. Put more strikingly — the combinatorics of transpositions in the symmetric group leads inexorably to the tautological ring of the moduli space of curves!

3.20. We return to our original motivation for discussing the ELSV formula: computing top intersections of ψ-classes on the moduli space of curves $\overline{\mathcal{M}}_{g,n}$. Fix g and n. As stated earlier, any given Hurwitz number may be readily computed (and this can be formalized elegantly in the language of generating functions). Thus any number of values of $P_{g,n}(\alpha_1, \ldots, \alpha_n)$ may be computed. However, we know that $P_{g,n}$ is a symmetric polynomial of known degree, and it is straightforward to show that one can determine the coefficients of a polynomial of known degree from enough values. In particular, from (9), the coefficients of the highest-degree terms in $P_{g,n}$ are precisely the top intersections of ψ-classes.

This is a powerful perspective. As an important example, Okounkov and Pandharipande used the ELSV formula to prove Witten's conjecture.

3.21. Back to Faber-type conjectures.

This concludes our discussion of Faber-type conjectures for $\overline{\mathcal{M}}_{g,n}$. I have two more remarks about Faber-type conjectures. The first is important, the second a side-remark.

3.22. *Faber's intersection number conjecture on $\mathcal{M}_g$, take two.* We define the moduli space of n-pointed genus g curves with "rational tails", denoted $\mathcal{M}^{rt}_{g,n}$, as follows. We define $\mathcal{M}^{rt}_{g,n}$ as the dense open subset of $\overline{\mathcal{M}}_{g,n}$ parametrizing pointed nodal curves where one component is nonsingular of genus g (and the remaining components form trees of genus 0 curves sprouting from it — hence the phrase "rational tails"). If $g > 1$, then $\mathcal{M}^{rt}_{g,n} = \pi^{-1}(\mathcal{M}_g)$, where $\pi : \overline{\mathcal{M}}_{g,n} \to \overline{\mathcal{M}}_g$ is the forgetful morphism. Note that $\mathcal{M}^{rt}_g = \mathcal{M}_g$.

We may restate Faber's intersection number conjecture (for $\mathcal{M}_g$) in terms of this moduli space. By our redefinition of the tautological ring on $\mathcal{M}_g$ in §3.3 (Definition 3.9, using also Faber's constructions of §3.12), the "top intersections" are determined by $\pi_*\psi_1^{a_1}\cdots\psi_n^{a_n}$ (where $\pi : \mathcal{M}^{rt}_{g,n} \to \mathcal{M}_g$) for $\sum a_i = g-2+n$.

Then Faber's intersection number conjecture translates to the following.

3.23. *Faber's intersection number conjecture (take two). — If all $a_i > 1$, then*

$$\psi_1^{a_1}\cdots\psi_n^{a_n} = \frac{(2g-3+n)!(2g-1)!!}{(2g-1)!\prod_{j=1}^{n}(2a_j-1)!!}[\textit{generator}] \quad \textit{for} \sum a_i = g-2+n$$

where $[\textit{generator}] = \kappa_{g-2} = \pi_*\psi_1^{g-1}$.

(This reformulation is also due to Faber.) This description is certainly more beautiful than the original one (4), which suggests that we are closer to the reason for it to be true.

3.24. The other conjectures of Faber were extended to $\mathcal{M}^{rt}_{g,n}$ by Pandharipande [P, Conj. 1].

3.25. *Remark: Faber-type conjectures for curves of compact type.* Based on the cases of $\mathcal{M}_g$ and $\mathcal{M}_{g,n}$, Faber and Pandharipande made another conjecture for curves of "compact type". A curve is said to be of *compact type* if its Jacobian is compact, or equivalently if its dual graph has no loops, or equivalently, if the curve has no nondisconnecting nodes. Define $\mathcal{M}^c_{g,n} \subset \overline{\mathcal{M}}_{g,n}$ to be the moduli space of curves of compact type. It is $\overline{\mathcal{M}}_{g,n}$ minus an irreducible divisor, corresponding to singular curves with one irreducible component (called Δ_0, although we will not use this notation).

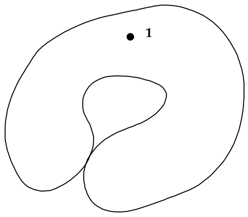

FIGURE 5. The curve corresponding to the point $\delta_0 \in \overline{\mathcal{M}}_{1,1}$

3.26. *Conjecture (Faber-Pandharipande* [FabP1, Spec. 2], [P, Conj. 1]. — $R^*(\mathcal{M}_g^c)$ *is a Poincaré duality ring of dimension* $2g-3$.

Again, this has a *vanishing/socle* part and a *perfect pairing* part. There is something that can be considered the corresponding *intersection number* part, Pandharipande and Faber's λ_g theorem [FabP2].

We will later (§4.7) give a proof of the vanishing/socle portion of the conjecture, that $R^i(\mathcal{M}_g^c) = 0$ for $i > 2g-3$, and is 1-dimensional if $i = 2g-3$. The perfect pairing part is essentially completely open.

3.27. Other relations in the tautological ring.

We have been concentrating on top intersections in the tautological ring. I wish to discuss more about other relations (in smaller codimension) in the tautological ring.

In genus 0, as stated earlier (§3.4), all classes on $\overline{\mathcal{M}}_{g,n}$ are generated by the strata, and the only relations among them are the cross-ratio relations. We have also determined the ψ-classes in terms of the boundary classes.

In genus 1, we can verify that ψ_1 can be expressible in terms of boundary strata. On $\overline{\mathcal{M}}_{1,1}$, if the boundary point is denoted δ_0 (the class of the nodal elliptic curve shown in Figure 5), we have shown $\psi_1 = \delta_0/12$. (Reason: we proved it was true on a finite cover, in the course of showing that $\int_{\overline{\mathcal{M}}_{1,1}} \psi_1 = 1/24$.) We know how to pull back ψ-classes by forgetful morphisms, so we can now verify the following.

Exercise. Show that in the cohomology group of $\overline{\mathcal{M}}_{1,n}$, ψ_i is equivalent to a linear combination of boundary divisors. (Hint: use the Comparison Lemma 3.6.)

3.28. *Slightly trickier exercise.* Use the above to show that the tautological ring in genus 1 is generated (as a group) by boundary classes. (This fact was promised in §3.4.)

In genus 2, this is no longer true: ψ_1 is not equivalent to a linear combination of boundary strata on $\overline{\mathcal{M}}_{2,1}$. However, in 1983, Mumford showed that ψ_1^2 (on $\overline{\mathcal{M}}_{2,1}$) is a combination of boundary strata ([Mu], see also [Ge2, eqn. (4)]); in 1998, Getzler showed the same for $\psi_1\psi_2$ (on $\overline{\mathcal{M}}_{2,2}$) [Ge2]. These two results can be used to show that on $\overline{\mathcal{M}}_{2,n}$, all

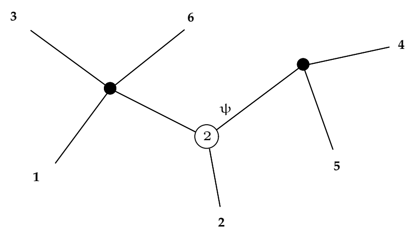

FIGURE 6. A class on $\overline{\mathcal{M}}_{2,6}$ — a codimension 1 class on a boundary stratum, constructed using ψ_1 on $\overline{\mathcal{M}}_{2,3}$ and gluing morphisms

tautological classes are linear combinations of strata, and classes "constructed using ψ_1 on $\overline{\mathcal{M}}_{2,1}$". Figure 6 may help elucidate what classes we mean — they correspond to dual graphs, with at most one marking ψ on an edge incident to one genus 2 component. The class in question is defined by gluing together the class of ψ_i on $\overline{\mathcal{M}}_{2,\nu}$ corresponding to that genus 2 component (where ν is the valence, and i corresponds to the edge labeled by ψ) with the fundamental classes of the $\overline{\mathcal{M}}_{0,\nu_j}$'s corresponding to the other vertices. The question then arises: what are the relations among these classes? On top of the cross-ratio and Getzler relation, there is a new relation due to Belorousski and Pandharipande, in codimension 2 on $\overline{\mathcal{M}}_{2,3}$ [BP]. We do not know if these three relations generate all the relations. (All the genus 2 relations mentioned in this paragraph are given by explicit formulas, although they are not pretty to look at.)

In general genus, the situation should get asymptotically worse as g grows. However, there is a general statement that can be made:

3.29. *Getzler's conjecture* [Ge2, footnote 1] *(Ionel's theorem* [I1]*). — If $g > 0$, all degree g polynomials in ψ-classes vanish on $\mathcal{M}_{g,n}$ (hence live on the boundary of $\overline{\mathcal{M}}_{g,n}$).*

We will interpret this result as a special case of a more general result (Theorem $\star$), in §4.3. In keeping with the theme of this article, the proof will be Gromov-Witten theoretic.

3.30. Y.-P. Lee's Invariance Conjecture. There is another general statement that may well give *all* the relations in *every* genus: Y.-P. Lee's Invariance conjecture. It is certainly currently beyond our ability to either prove or disprove it, although the first part is already a theorem (Theorem 3.31). Lee's conjecture is strongly motivated by Gromov-Witten theory.

Before we state the conjecture, we discuss the consequences and evidence. All of the known relations in the tautological rings are consequences of the conjecture. For example, the genus 2 implications are shown by Arcara and Lee in [ArcL1]. They then predicted a *new* relation in $\overline{\mathcal{M}}_{3,1}$ in [ArcL2]. Simultaneously and independently, this relation was proved by Kimura and X. Liu [KL]. This seems to be good evidence for the conjecture being true.

More recently, the methods behind the conjecture have allowed Lee to turn these predictions into proofs, *not conditional on the truth of the conjecture* [Lee2]. Thus for example Arcara and Lee's work yields a proof of the new relation on $\overline{\mathcal{M}}_{3,1}$.

We now give the statement. The conjecture is most naturally expressed in terms of the tautological rings of *possibly-disconnected* curves. The definition of a stable possibly-disconnected curve is the same as that of a stable curve, except the curve is not required to be connected. We denote the moduli space of n-pointed genus g possibly-disconnected curves by $\overline{\mathcal{M}}^{\bullet}_{g,n}$. The reader can quickly verify that our discussion of the moduli space of curves carries over without change if we consider possibly-disconnected curves. For example, $\overline{\mathcal{M}}^{\bullet}_{g,n}$ is nonsingular and pure-dimensional of dimension $3g-3+n$ (although not in general irreducible). It contains $\overline{\mathcal{M}}_{g,n}$ as a component, so any statements about $\overline{\mathcal{M}}^{\bullet}_{g,n}$ will imply statements about $\overline{\mathcal{M}}_{g,n}$. Note that the disjoint union of two curves of arithmetic genus g and h is a curve of arithmetic genus $g+h-1$: Euler characteristics add under disjoint unions. Note also that a possibly-disconnected marked curve is stable if and only if all of its connected components are stable.

Exercise. Show that $\overline{\mathcal{M}}^{\bullet}_{-1,6}$ is a union of $\binom{6}{3}/2$ points — any 6-pointed genus -1 stable curve must be the disjoint union of two $\mathbb{P}^1$'s, with 3 of the 6 labeled points on each component.

Exercise. Show that any component of $\overline{\mathcal{M}}^{\bullet}_{g,n}$ is the quotient of a product of $\overline{\mathcal{M}}_{g',n'}$'s by a finite group.

Tautological classes are generated by classes corresponding to a dual graph, with each vertex (of genus g and valence n, say) labeled by some cohomology class on $\overline{\mathcal{M}}^{\bullet}_{g,n}$ (possibly the fundamental class); call this a decorated dual graph. (We saw an example of a decorated dual graph in Figure 6. Note that ψ-classes will always be associated to some half edge.) Decorated dual graphs are not required to be connected. If Γ is a decorated dual graph (of genus g with n tails, say), let $\dim\Gamma$ be the dimension of the corresponding class in $A_*(\overline{\mathcal{M}}^{\bullet}_{g,n})$.

For each positive integer l, we will describe a linear operator $\mathfrak{r}_l$ that sends formal linear combinations of decorated dual graphs to formal linear combinations of decorated dual graphs. It is homogeneous of degree $-l$: it sends (dual graphs corresponding to) dimension k classes to (dual graphs corresponding to) dimension $k-l$ classes.

We now describe its action on a single decorated dual graph Γ of genus g with n marked points (or half-edges), labeled 1 through n. Then $\mathfrak{r}_l(\Gamma)$ will be a formal linear combination of other graphs, each of genus $g-1$ with $n+2$ marked points.

There are three types of contributions to $\mathfrak{r}_l(\Gamma)$. (In each case, we discard any graph that is not stable.)

1. Edge-cutting. There are two contributions for each *directed* edge, i.e. an edge with chosen starting and ending point. (Caution: there are two possible directions for each edge in general, except for those edges that are "loops", connecting a single vertex to itself. In this case, both directions are considered the same.) We cut the edge, regarding the two half-edges as "tails", or marked points. The starting half-edge is labeled $n+1$, and the ending half-edge is labeled $n+2$. One summand will correspond to adding an extra decoration of ψ^l to point $n+1$. (In other words, ψ^l_{n+1} is multiplied by whatever cohomology class is already decorating that vertex.) A second summand will correspond to the adding an extra decoration of ψ^l to point $n+2$, and this summand appears with multiplicity $(-1)^{l-1}$.

2. Genus reduction For each vertex we produce l graphs as follows. We reduce the genus of the vertex by 1, and add two new tails to this vertex, labelled $n+1$ and $n+2$; we decorate them with ψ^m and ψ^{l-1-m} respectively, where $0 \leq m \leq l-1$. Each such graph is taken with multiplicity $(-1)^{m+1}$.

3. Vertex-splitting. For each vertex, we produce a number of graphs as follows. We split the vertex into two. The first new vertex is given the tail $n+1$, and the second is given the tail $n+2$. The two new tails are decorated by ψ^m and ψ^{l-1-m} respectively, where $0 \leq m \leq l-1$. We then take one such graph for each choice of splitting of the genus $g = g_1 + g_2$ and partitioning of the other incident edges. Each such graph is taken with multiplicity $(-1)^{m+1}$.

Then $\mathfrak{r}_l(\Gamma)$ is the sum of the above summands. Observe that when l is odd (resp. even), the result is symmetric (resp. anti-symmetric) in the labels $n+1$ and $n+2$.

By linearity, this defines the action of $\mathfrak{r}_l$ on any linear combination of directed graphs.

3.31. *Y.P. Lee's Invariance Theorem* [Lee2]. — *If* $\sum c_i \Gamma_i = 0$ *holds in* $A^*(\overline{\mathcal{M}}^\bullet_{g,n})$*, then* $\mathfrak{r}_l(\sum c_i \Gamma_i) = 0$ *in* $A^*(\overline{\mathcal{M}}_{g-1,n+2})$.

This was invariance conjecture 1 of [Lee1]. It gives a necessary condition for a tautological class to be zero. Hence for example it can be used to determine the coefficients of a tautological equation, if we already know there is one by other means. The theorem also implies that $\mathfrak{r}_l$ is well-defined at the level of tautological rings (i.e. compatible with descending). The converse of Theorem 3.31 is also conjectured to be true, and would be a sufficient condition for a candidate tautological equation to hold true:

3.32. *Y.-P. Lee's Invariance Conjecture* [Lee1, Conj. 2]. — *If* $\sum c_i \Gamma_i$ *has* **positive** *pure dimension, and* $\mathfrak{r}_l(\sum c_i \Gamma_i) = 0$ *in* $A^*(\overline{\mathcal{M}}_{g-1,n+2})$*, then* $\sum c_i \Gamma_i = 0$ *holds in* $A^*(\overline{\mathcal{M}}^\bullet_{g,n})$.

Theorem 3.31 and Conjecture 3.32 can be used to produce tautological equations inductively! The base case is when $\dim \overline{\mathcal{M}}^\bullet_{g,n} = 0$, which is known: we will soon show (§4.6) that $R_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$. (We write $R^{3g-3+n}(\overline{\mathcal{M}}_{g,n})$ as $R_0(\overline{\mathcal{M}}_{g,n})$ to remind the reader that the statement is about tautological 0-cycles.) Hence dimension 0 tautological classes on $\overline{\mathcal{M}}_{g,n}$ are determined by their degree (and dimension 0 tautological classes on $\overline{\mathcal{M}}^\bullet_{g,n}$ are determined by their degree on each connected component). Note that the algorithm is a finite process: the dimension l relations on $\overline{\mathcal{M}}_{g,n}$ or $\overline{\mathcal{M}}^\bullet_{g,n}$ produced by this algorithm are produced after a finite number of steps.

Even more remarkably, this seems to produce *all* tautological relations:

3.33. *Y.-P. Lee's Invariance Conjecture, continued* [Lee1, Conj. 3]. — *Conjecture 3.32 will produce* **all** *tautological equations inductively.*

A couple of remarks are in order. Clearly this is a very combinatorial description. It was dictated by Gromov-Witten theory, as explained in [Lee1]. In particular, it uses the fact that all tautological equations are invariant under the action of lower triangular subgroups of the twisted loop groups, and proposes that they are the only equations invariant in this way.

In order to see the magic of this conjecture in action, and to get experience with the $\mathfrak{r}_l$ operators, it is best to work out an example. The simplest dimension 1 relation is the following.

Exercise. Show that the pullback of the (dimension 0) cross-ratio relation (§3.4) on $\overline{\mathcal{M}}_{0,4}$ to a (dimension 1) relation on $\overline{\mathcal{M}}_{0,5}$ is implied by the Invariance Conjecture. (Some rather beautiful cancellation happens.)

3.34. *Final remarks on relations in the tautological ring.* This continues to be an active area of research. We point out for example Arcara and Sato's recent article [ArcS] using localization in Gromov-Witten theory to compute the class $\sum_{k=0}^{g}(-1)^{g-k}\psi_1^k\lambda_{g-k}$ explicitly as a sum of boundary classes on $\overline{\mathcal{M}}_{g,1}$.

4. A BLUNT TOOL: THEOREM $\star$ AND CONSEQUENCES

We now describe a blunt tool from which much of the previously described structure of the tautological ring follows. Although it is statement purely about the stratification of the moduli space of curves, we will see (§6.3) that it is proved via Gromov-Witten theory.

4.1. *Theorem* $\star$ [GrV3]. — *Any tautological class of codimension* i *is trivial away from strata satisfying*

$$\boxed{\#\textit{ genus } 0 \textit{ vertices } \geq i-g+1.}$$

(Recall that the genus 0 vertices correspond to components of the curve with geometric genus 0.)

More precisely, any tautological class is zero upon restriction to the (large) open set corresponding to the open set corresponding to

$$\#\text{ genus } 0 \text{ vertices } < i-g+1.$$

Put another way: given any tautological class of codimension i, you can move it into the set of curves with at least $i-g+1$ genus 0 components. A third formulation is that the tautological classes of codimension i are pushed forward from classes on the locus of curves with at least $i - g + 1$ genus 0 components.

We remark that this is false for the Chow ring as a whole — this is fundamentally a statement about tautological classes.

We will discuss the proof in §6.3, but first we give consequences. There are in some sense four morals of this result.

First, this is the fundamental geometry behind many of the theorems we have been discussing. We will see that they follow from Theorem $\star$ by straightforward combinatorics. As a sign of this, we will often get strengthenings of what was known or conjectured previously.

Second, this suggests the potential importance of a filtration of the moduli space by the number of genus 0 components. It would be interesting to see if this filtration really is fundamental, for example if it ends up being relevant in understanding the moduli space of curves in another way. So far this has not been the case.

Third, as we will see from the proof, once one knows a clean statement of what one wants to prove, the proof is relatively straightforward, at least in outline.

And fourth, the proof will once again show the centrality of Gromov-Witten theory to the study of the moduli of curves.

4.2. Consequences of Theorem $\star$.

We begin with a warm-up example.

4.3. *Theorem $\star$ implies Getzler's conjecture 3.29 (Ionel's theorem).* Any degree g monomial is a codimension g tautological class, which vanishes on the open set of $\overline{\mathcal{M}}_{g,n}$ corresponding to curves with no genus 0 components. If $g > 0$, this is non-empty and includes $\mathcal{M}_{g,n}$.

In particular: (1) we get a proof of Getzler's conjecture; (2) we see that more classes vanish on this set — all tautological classes of degree at least g, not just polynomials in the ψ-classes; (3) we observe that the classes vanish on a bigger set than $\mathcal{M}_{g,n}$, and that what is relevant is not the smoothness of the curves, but the fact that they have no genus 0 components. (4) This gives a moral reason for Getzler's conjecture not to hold in genus 0.

4.4. *Theorem $\star$ implies the first part of Looijenga's Theorem (Faber's vanishing conjecture).* Recall (§3.2) that Looijenga's Theorem is part of the "vanishing" part of Faber's conjectures:

4.5. *Theorem* [Lo]. — *We have* $R^i(\mathcal{M}_g) = 0$ *for* $i > g - 2$, $\dim R^{g-2}(\mathcal{M}_g) \leq 1$.

We will show that Theorem $\star$ implies the first part now; we will show the second part as a consequence of Theorem 7.10.

First, if the codimension of a tautological class is greater than or equal to g, then we get vanishing on the open set where there are no genus 0 components, so we get vanishing for the same reason as in Getzler's conjecture.

The case of codimension $g-1$ is more subtle. From the definition of the tautological ring, tautological classes are obtained by taking ψ-classes, and multiplying, gluing, and pushing forward by forgetful morphisms. Now on $\mathcal{M}_g = \mathcal{M}_{g,0}$, there are no ψ-classes and no boundary strata, so by the definition of the tautological ring, all codimension $g-1$ tautological classes on $\mathcal{M}_{g,0}$ are pushed forward from tautological classes on $\mathcal{M}_{g,1}$, which are necessarily of codimension g. These also vanish by Theorem $\star$ by the same argument as before.

As before, one can say more:

Exercise. Extend this argument to the moduli space of curves with rational tails $\mathcal{M}_{g,n}^{rt}$. (First determine the dimension of the conjectural Poincaré duality ring!)

The Faber-type conjecture for this space was mentioned in §3.24. I should point out that I expect that Looijenga's proof extends to this case without problem, but I haven't checked.

4.6. *Theorem $\star$ implies the socle part of the Hain-Looijenga-Faber-Pandharipande conjecture 3.11 on $\overline{\mathcal{M}}_{g,n}$.* Recall the socle part of the Hain-Looijenga-Faber-Pandharipande conjecture 3.11, that $R_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$.

We show how this is implied by Theorem $\star$. This was first shown in [GrV2], which can be seen as a first step toward the statement and proof of Theorem $\star$.

Our goal is to show that all tautological 0-cycles are commensurate, and that one of them is non-zero. Clearly the latter is true, as the class of a 0-dimensional stratum (a point) is tautological, and is non-zero as it has non-zero degree, so we concentrate on the first statement.

By Theorem $\star$, any dimension 0 tautological class is pushed forward from the locus of curves with at least $(3g-3+n)-g+1 = 2g-2+n$ genus 0 components.

Exercise. Show that the only stable dual graphs with $2g-2+n$ genus 0 components have all vertices genus 0 and trivalent. Show that these are the 0-dimensional strata. (See Figure 7 for the 0-dimensional strata on $\overline{\mathcal{M}}_{1,2}$.)

Hence $R_0(\overline{\mathcal{M}}_{g,n})$ is generated by this finite number of points. It remains to show that any two of these points are equivalent in the Chow ring. A geometric way of showing this is by observing that all points in $\overline{\mathcal{M}}_{0,N}$ are equivalent in the Chow ring, and that our 0-dimensional strata are in the image of $\overline{\mathcal{M}}_{0,2g+n}$ under g gluing morphisms. A more

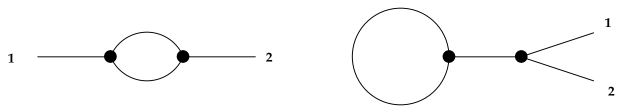

FIGURE 7. The 0-dimensional strata on $\overline{\mathcal{M}}_{1,2}$ — notice that all vertices are genus 0 and trivalent, and that there are $2g - 2 + n$ of them

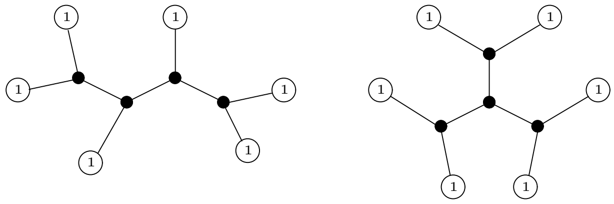

FIGURE 8. The two 0-pointed genus 6 stable trees with at least 4 genus 0 vertices

combinatorial way of showing this is by showing that each 1-dimensional stratum is isomorphic to $\mathbb{P}^1$, and that any two 0-dimensional strata can be connected by a chain of 1-dimensional strata.

Exercise. Complete one of these arguments.

As in the earlier applications of Theorem $\star$ too: we can verify the perfect pairing conjecture in codimension 1 and probably 2 (although Tom Graber and I haven't delved too deeply into 2). This is combinatorially more serious, but not technically hard.

4.7. *Theorem $\star$ implies the Faber-Pandharipande vanishing/socle conjecture on curves of compact type.* We now show the "vanishing/socle part" of the Faber-type conjecture for curves of compact type (Faber-Pandharipande Conjecture 3.26).

First, suppose that $i > 2g - 3$. We will show that $R^i(\mathcal{M}_g^c) = 0$. By Theorem $\star$, any such tautological class vanishes on the open set where there are at most $i - g + 1 > g - 2$ genus 0 vertices. Then our goal follows from the next exercise.

Exercise. Show that any genus g (0-pointed) stable graph that is a tree has at most $g - 2$ genus 0 vertices. Moreover, if equality holds, then each vertex is either genus 1 of valence 1, or genus 0 of valence 3. (Examples when $g = 6$ are given in Figure 8.)

Next, if $i = 2g - 3$, then our codimension $2g - 3$ (hence g) class is pushed forward from strata of the form described in the previous exercise. But each stratum has dimension g, so the tautological class must be a linear combination of fundamental classes of such strata.

Furthermore, any two such strata are equivalent (in cohomology, or even in the Chow ring) by arguments analogous to either of those we used for $\overline{\mathcal{M}}_{g,n}$.

Thus we have shown that $R^{2g-3}(\mathcal{M}_g^c)$ is generated by the fundamental class of a single such stratum. It remains to show that this is non-zero. This argument is short, but requires a little more background than we have presented. (For the experts: it suffices to show that $\lambda_g \neq 0$ on this stratum $\mathcal{M}_\Gamma$. We have a cover $\pi \overline{\mathcal{M}}_{1,1}^g \to \mathcal{M}_\Gamma$ via gluing morphisms, and the pullback of the Hodge bundle splits into the Hodge bundles of each of the g elliptic curves. Thus $\pi^*\lambda_g$ is the product of the λ_1-classes on each factor, so $\deg \pi^*\lambda_g = (\int_{\overline{\mathcal{M}}_{1,1}} \lambda_1)^g = 1/24^g \neq 0$.)

As always, Theorem $\star$ gives extra information. (1) This argument extends to curves of compact type with points. (2) We can now attack part of the Poincaré duality portion of the conjecture. (3) We get an explicit generator of $R^{2g-3}(\mathcal{M}_g^c)$ (a stratum of a particular form, e.g. Figure 8).

4.8. *Theorem $\star$ helps determine the tautological ring in low dimension.* In the course of proving $R_0(\overline{\mathcal{M}}_{g,n}) \cong \mathbb{Q}$, we showed that $R_0(\overline{\mathcal{M}}_{g,n})$ was generated by 0-strata. A similar argument shows that $R_i(\overline{\mathcal{M}}_{g,n})$ generated by boundary strata for $i = 1, 2$ (where $R_i(\overline{\mathcal{M}}_{g,n}) := R^{3g-3+n-i}(\overline{\mathcal{M}}_{g,n})$). (We are already aware that this will not extend to $i = 3$, as ψ_1 on $\overline{\mathcal{M}}_{2,1}$ is not a linear combination of fundamental classes of strata.)

In general, Theorem $\star$ implies that in order to understand tautological classes in dimension up to i, you need only understand curves of genus up to $(i+1)/2$, with not too many marked points.

The moral of this is that the "top" (lowest-codimension) part of the tautological ring used to be considered the least mysterious (given the definition of the tautological ring, it is easy to give generators), and the bottom was therefore the most mysterious. Now the situation is the opposite. For example, in codimension 3, we can describe the generators of the tautological ring, but we have no idea what the relations are. However, we know exactly what the tautological ring looks like in *dimension* 3.

4.9. *Exercise.* Use Theorem $\star$ and a similar argument to show that the tautological groups of $\overline{\mathcal{M}}_{0,n}$ and $\overline{\mathcal{M}}_{1,n}$ are generated by boundary strata.

4.10. *Additional consequences.* For many additional consequences of Theorem $\star$, see [GrV3]. For example, we recover *Diaz' theorem* ($\mathcal{M}_g$ contains no complete subvarieties of $\dim > g-2$), as well as generalizations and variations such as: $\overline{\mathcal{M}}_{g,n}^c$ contains no complete subvarieties of $\dim > 2g-3+n$.

The idea behind the proof of Theorem $\star$ is rather naive. But before we can discuss it, we will have to finally enter the land of Gromov-Witten theory, and define stable relative maps to $\mathbb{P}^1$, which we will interpret as a generalization of the notion of a branched cover.

5. Stable relative maps to $\mathbb{P}^1$ and relative virtual localization

We now discuss the theory of *stable relative maps*, and "virtual" localization on their moduli space (*relative virtual localization*). We will follow J. Li's algebro-geometric definition of stable relative maps [Li1], and his description of their obstruction theory [Li2], but we point out earlier definitions of stable relative maps in the differentiable category due to A.-M. Li and Y. Ruan [LR], and Ionel and Parker [IP1, IP2], and Gathmann's work in the algebraic category in genus 0 [Ga]. We need the algebraic category for several reasons, most importantly because we will want to apply virtual localization.

Stable relative maps are variations of the notion of stable maps, and the reader may wish to become comfortable with that notion first. (Stable maps are discussed in Abramovich's article in this volume, for example.)

We are interested in the particular case of stable relative maps to $\mathbb{P}^1$, relative to at most two points, so we will define stable relative maps only in this case. For concreteness, we define stable maps to $X = \mathbb{P}^1$ relative to one point ∞; the case of zero or two points is the obvious variation on this theme. Such a stable relative map to $(\mathbb{P}^1, \infty)$ is defined as follows. We are given the data of a degree d of the map, a genus g of the source curve, a number m of marked points, and a partition $d = \alpha_1 + \cdots + \alpha_n$, which we write $\alpha \vdash d$.

Then a relative map is the following data:

- a morphism f_1 from a nodal $(m+n)$-pointed genus g curve $(C, p_1, \ldots, p_m, q_1, \ldots, q_n)$ (where as usual the p_i and q_j are distinct nonsingular points) to a chain of $\mathbb{P}^1$'s, $T = T_0 \cup T_1 \cup \cdots \cup T_t$ (where T_i and T_{i+1} meet), with a point $\infty \in T_t - T_{t-1}$. Unfortunately, there are two points named ∞. We will call the one on X, ∞_X, and the one on T, ∞_T, whenever there is any ambiguity.
- A projection $f_2 : T \to X$ contracting T_i to ∞_X (for $i > 0$) and giving an isomorphism from $(T_0, T_0 \cap T_1)$ (resp. (T_0, ∞)) to X if $t > 0$ (resp. if $t = 0$). Denote $f_2 \circ f_1$ by f.
- We have an equality of divisors on C: $f_1^* \infty_T = \sum \alpha_i q_i$. In particular, $f_1^{-1} \infty_T$ consists of nonsingular (marked) points of C.
- The preimage of each node n of T is a union of nodes of C. At any such node n' of C, the two branches map to the two branches of n, and their orders of branching are the same. (This is called the *predeformability* or *kissing* condition.)

If follows that the degree of f_1 is d on each T_i. An *isomorphism* of two such maps is a commuting diagram

$$\begin{array}{ccc}
(C, p_1, \ldots, p_m, q_1, \ldots, q_n) & \xrightarrow{\sim} & (C', p'_1, \ldots, p'_m, q'_1, \ldots, q'_n) \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_1} \\
(T, \infty_T) & \xrightarrow{\sim} & (T, \infty_T) \\
\downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_2} \\
(X, \infty_X) & \xrightarrow{=} & (X, \infty_X)
\end{array}$$

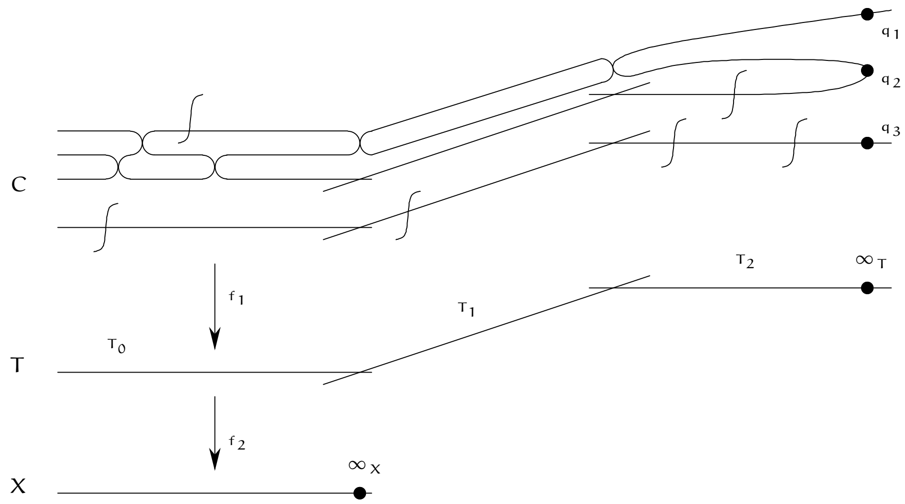

FIGURE 9. An example of a stable relative map

where all horizontal morphisms are isomorphisms, the bottom (although not necessarily the middle!) is an equality, and the top horizontal isomorphism sends p_i to p'_i and q_j to q'_j. Note that the middle isomorphism must preserve the isomorphism of T_0 with X, and is hence the identity on T_0, but for $i > 0$, the isomorphism may not be the identity on T_i.

This data of a relative map is often just denoted f, with the remaining information left implicit.

We say that f is *stable* if it has finite automorphism group. This corresponds to the following criteria.

- Any f_1-contracted geometric genus 0 component has at least 3 "special points" (node branches or marked points).
- Any f_1-contracted geometric genus 1 component has at least 1 "special point".
- If $0 < i < t$ (resp. $0 < i = t$), then not every component mapping to T_i is of the form $[x; y] \to [x^q; y^q]$, where the coordinates on the target are given by $[0; 1] = T_i \cap T_{i-1}$ and $[1; 0] = T_i \cap T_{i+1}$ (resp. $[1; 0] = \infty$).

(The first two conditions are the same as for stable maps. The third condition is new.) A picture of a stable relative map is given in Figure 9.

Thus we have some behavior familiar from the theory of stable maps: we can have contracted components, so long as they are "stable", and don't map to any nodes of T, or to ∞_T. We also have some new behavior: the target X can "sprout" a chain of $\mathbb{P}^1$'s at ∞_X. Also, the action of $\mathbb{C}^*$ on the map via the action on a component T_i ($i > 0$) that preserves the two "special points" of T_i (the intersections with T_{i-1} and T_{i+1} if $i < t$, and the intersection with T_{i-1} and ∞ if $i = t$) is considered to preserve the stable map. For example, Figure 10 shows two isomorphic stable maps.

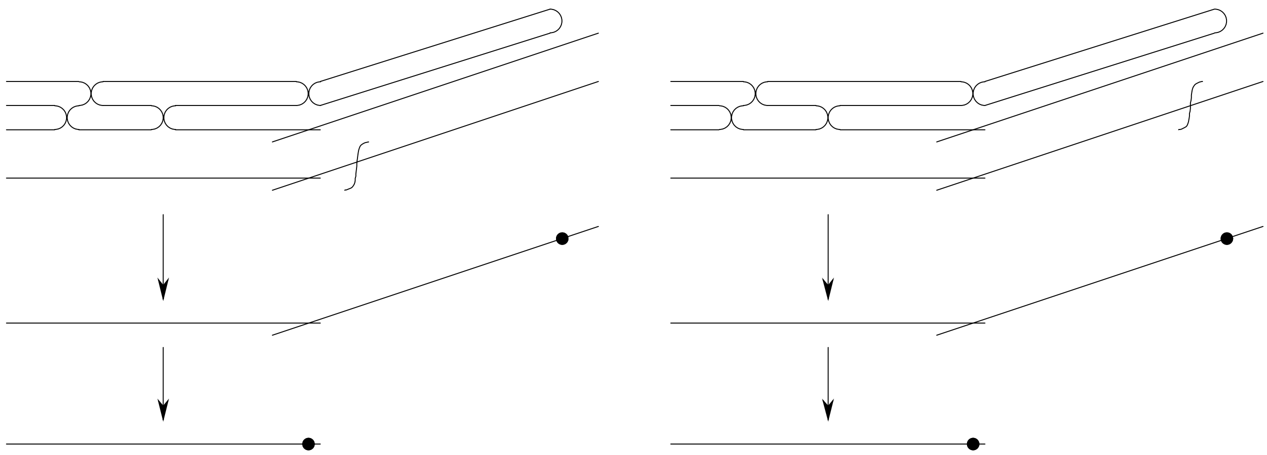

FIGURE 10. Two isomorphic stable relative maps

There is a compact moduli space (Deligne-Mumford stack) for stable relative maps to $\mathbb{P}^1$, denoted $\overline{\mathcal{M}}_{g,m,\alpha}(\mathbb{P}^1, d)$. (In order to be more precise, I should tell you the definition of a family of stable relative maps parametrized by an arbitrary base, but I will not do so.) In what follows, $m = 0$, and that subscript will be omitted. (More generally, stable relative maps may be defined with $\mathbb{P}^1$ replaced by any smooth complex projective variety, and the point ∞ replaced by any smooth divisor D on X. The special case $D = \emptyset$ yields Kontsevich's original space of stable maps.)

Unfortunately, the space $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)$ is in general terribly singular, and not even equidimensional.

Exercise. Give an example of such a moduli space with two components of different dimensions. (Hint: use contracted components judiciously.)

However, it has a component which we already understand well, which corresponds to maps from a smooth curve, which is a branched cover of $\mathbb{P}^1$. Such curves form a moduli space $\mathcal{M}_{g,\alpha}(\mathbb{P}^1, d)$ of dimension corresponding to the "expected number of branch points distinct from ∞", which we may calculate by the Riemann-Hurwitz formula (2) to be

$$r = 2g - 2 + n + d. \tag{10}$$

We have seen this formula before, (6).

Exercise. Verify (10).

These notions can be readily generalized, for example to stable relative maps to $\mathbb{P}^1$ relative to two points (whose moduli space is denoted $\overline{\mathcal{M}}_{g,\alpha,\beta}(\mathbb{P}^1, d)$), or to no points (otherwise known as the stable maps to $\mathbb{P}^1$; this moduli space is denoted $\overline{\mathcal{M}}_g(\mathbb{P}^1, d)$).

Exercise. Calculate $\dim \mathcal{M}_{g,\alpha,\beta}(\mathbb{P}^1, d)$ (where α has m parts and β has n parts) and $\dim \mathcal{M}_g(\mathbb{P}^1, d)$.

5.1. *Stable relative maps with possibly-disconnected source curve.* Recall that by our (non-standard) definition, nodal curves are connected. It will be convenient, especially when discussing the degeneration formula, to consider curves without this hypothesis. Just as our discussion of (connected) stable curves generalized without change to (possibly-disconnected) stable curves (see §3.30), our discussion of (relatively) stable maps from connected curves generalizes without change to "(relatively) stable maps from possibly-disconnected curves". Let $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)^\bullet$ be the space of stable relative maps from possibly-disconnected curves (to $\mathbb{P}^1$, of degree d, etc.). Warning: this is *not* in general the quotient of a product of $\overline{\mathcal{M}}_{g',\alpha'}(\mathbb{P}^1, d')$'s by a finite group.

5.2. The virtual fundamental class.

There is a natural homology (or Chow) class on $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)$ of dimension $r = \dim \mathcal{M}_{g,\alpha}(\mathbb{P}^1, d)$ (cf. (10)), called the *virtual fundamental class* $[\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)]^{\mathrm{virt}} \in A_r(\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)])$, which is obtained from the deformation-obstruction theory of stable relative maps, and has many wonderful properties. The virtual fundamental class agrees with the actual fundamental class on the open subset $\mathcal{M}_{g,\alpha}(\mathbb{P}^1, d)$. The most difficult part of dealing with the moduli space of stable relative maps is working with the virtual fundamental class.

Aside: relative Gromov-Witten invariants. In analogy with usual Gromov-Witten invariants, one can define *relative Gromov-Witten invariants* by intersecting natural cohomology classes on the moduli space with the virtual fundamental class. More precisely, one multiplies (via the cup/cap product) the cohomology classes with the virtual fundamental class, and takes the degree of the resulting zero-cycle. One can define ψ-classes and λ-classes in the same way as before, and include these in the product. When including ψ-classes, the numbers are often called *descendant relative invariants*; when including λ-classes, the numbers are sometimes called *Hodge integrals*. For example, one can show that Hurwitz numbers are descendant relative invariants of $\mathbb{P}^1$. However, this point of view turns out to be less helpful, and we will not use the language of relative Gromov-Witten invariants again.

The virtual fundamental class behaves well under two procedures: degeneration and localization; we now discuss these.

5.3. The degeneration formula for the virtual fundamental class, following [Li2]**.**

We describe the degeneration formula in the case of stable maps to $\mathbb{P}^1$ relative to one point, and leave the cases of stable maps to $\mathbb{P}^1$ relative to zero or two points as straightforward variations for the reader. In this discussion, we will deal with possibly-disconnected curves to simplify the exposition.

Consider the maps to $\mathbb{P}^1$ relative to one point ∞, and imagine deforming the target so that it breaks into two $\mathbb{P}^1$'s, meeting at a node (with ∞ on one of the components). It turns out that the virtual fundamental class behaves well under this degeneration. The limit can be expressed in terms of virtual fundamental classes of spaces of stable relative maps to each component, relative to ∞ (for the component containing ∞), and relative to the node-branch (for both components).

Before we make this precise, we give some intuition. Suppose we have a branched cover $C \to \mathbb{P}^1$, and we deform the target into a union of two $\mathbb{P}^1$'s, while keeping the branch points away from the node; call the limit map $C' \to \mathbb{P}^1 \cup \mathbb{P}^1$. Clearly in the limit, away from the node, the cover looks just the same as it did before (with the same branching). At the node, it turns out that the branched covers of the two components must satisfy the kissing/predeformability condition. Say that the branching above the node corresponds to the partition $\gamma_1 + \cdots + \gamma_m$. By our discussion about Hurwitz numbers, as we have specified the branch points, there will be a finite number of such branched covers — we count branched covers of each component of $\mathbb{P}^1 \cup \mathbb{P}^1$, with branching corresponding to the partition γ above the node-branch; then we choose how to match the preimages of the node-branch on the two components (there are $\# \operatorname{Aut} \gamma$ such choices). It turns out that $\gamma_1 \cdots \gamma_m$ covers of the original sort will degenerate to each branched cover of the nodal curve of this sort. (Notice that if we were interested in connected curves C, then the inverse image of each component of $\mathbb{P}^1$ would not necessarily be connected, and we would have to take some care in gluing these curves together to get a connected union. This is the reason for considering possibly-disconnected components.)

Motivated by the previous paragraph, we give the degeneration formula. Consider the degeneration of the target $(\mathbb{P}^1, \infty) \rightsquigarrow \mathbb{P}^1 \cup (\mathbb{P}^1, \infty)$. Let (X, ∞) be the general target, and let (X', ∞) be the degenerated target. Let $(X_1, a_1) \cong (\mathbb{P}^1, \infty)$ denote the first component of X', where a_1 refers to the node-branch, and let $(X_2, a_2, \infty) \cong (\mathbb{P}^1, 0, \infty)$ denote the second component of X', where a_2 corresponds to the node branch. Then for each partition $\gamma_1 + \cdots + \gamma_m = d$, there is a natural map

$$\overline{\mathcal{M}}_{g_1,\gamma}(\mathbb{P}^1, d)^\bullet \times \overline{\mathcal{M}}_{g_2,\gamma,\alpha}(\mathbb{P}^1, d)^\bullet \to \overline{\mathcal{M}}_{g_1+g_2-m+1,\alpha}(X', d)^\bullet \tag{11}$$

obtained by gluing the points above a_1 to the corresponding points above a_2. The image of this map can be suitably interpreted as stable maps to X', satisfying the kissing condition, which can appear as the limit of maps to X. (We are obscuring a delicate issue here — we have not defined stable maps to a singular target such as X.) Then Li's degeneration formula states that the image of the product of the virtual fundamental classes in (11) is the limit of the virtual fundamental class of $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)^\bullet$, multiplied by $\gamma_1 \cdots \gamma_m$.

The main idea behind Li's proof is remarkably elegant, but as with any argument involving the virtual fundamental class, the details are quite technical.

If we are interested in connected curves, then there is a corresponding statement (that requires no additional proof): we look at the component of the moduli space on the right side of (11) corresponding to maps from connected source curves, and we look at just those components of the moduli spaces on the left side which glue together to give connected curves.

5.4. Relative virtual localization [GrV3].

The second fundamental method of manipulating virtual fundamental classes is by means of localization. Before discussing localization in our Gromov-Witten-theoretic context, we first quickly review localization in its original setting.

(A friendly introduction to equivariant cohomology is given in [HKKPTVVZ, Ch. 4], and to localization on the space of ordinary stable maps in [HKKPTVVZ, Ch. 27].)

Suppose Y is a complex projective manifold with an action by a torus $\mathbb{C}^*$. Then the fixed point locus of the torus is the union of smooth submanifolds, possibly of various dimensions. Let the components of the fixed locus be $Y_1, Y_2, \ldots$. The torus acts on the normal bundle N_i to Y_i. Then the Atiyah-Bott localization formula states that

$$[Y] = \sum_{\text{fixed}} [Y_i]/c_{\text{top}}(N_i) = \sum_{\text{fixed}} [Y_i]/e(N_i), \tag{12}$$

in the *equivariant homology* of Y (with appropriate terms inverted), where c_{top} (or the Euler class e) of a vector bundle denotes the top Chern class. This is a wonderfully powerful fact, and to appreciate it, you must do examples yourself. The original paper of Atiyah and Bott [AB] is beautifully written and remains a canonical source.

You can cap (12) with various cohomology classes to get 0-dimensional classes, and get an equality of numbers. But you can cap (12) with classes to get higher-dimensional classes, and get equality in cohomology (or the Chow ring). One lesson I want to emphasize is that this is a powerful thing to do. For example, in a virtual setting, in Gromov-Witten theory, localization is traditionally used to get equalities of numbers. We will also use equalities of numbers to prove the ELSV formula (8). However, using more generally equalities of classes will give us Theorem $\star$, and part of Faber's conjecture.

Localization was introduced to Gromov-Witten theory by Kontsevich in his groundbreaking paper [Ko2], in which he works on the space of genus zero maps to projective space, where the virtual fundamental class is the usual fundamental class (and hence there are no "virtual" technicalities). In the foundational paper [GrP], Graber and Pandharipande showed that the localization formula (12) works "virtually" on the moduli space of stable maps, where fundamental classes are replaced by virtual fundamental classes, and normal bundles are replaced by "virtual normal bundles". They defined the virtual fundamental class of a fixed locus, and the virtual normal bundle, and developed the machinery to deal with such questions.

There is one pedantic point that must be made here. The localization formula should reasonably be expected to work in great generality. However, we currently know it only subject to certain technical hypotheses. (1) The proof only works in the algebraic category. (2) In order to apply this machinery, the moduli space must admit a $\mathbb{C}^*$-equivariant locally closed immersion into an orbifold. (3) The virtual fundamental class of this fixed locus needs to be shown to arise from the $\mathbb{C}^*$-fixed part of the obstruction theory of the moduli space. It would be very interesting, and potentially important, to remove hypotheses (1) and (2).

The theory of virtual localization can be applied to our relative setting [GrV3]. (See [LLZ] for more discussion.) We now describe it in the case of interest to us, of maps to $\mathbb{P}^1$. Again, in order to understand this properly, you should work out examples yourself.

Fix a torus action on $\mathbb{P}^1$

$$\sigma \circ [x;y] = [\sigma x; y],$$

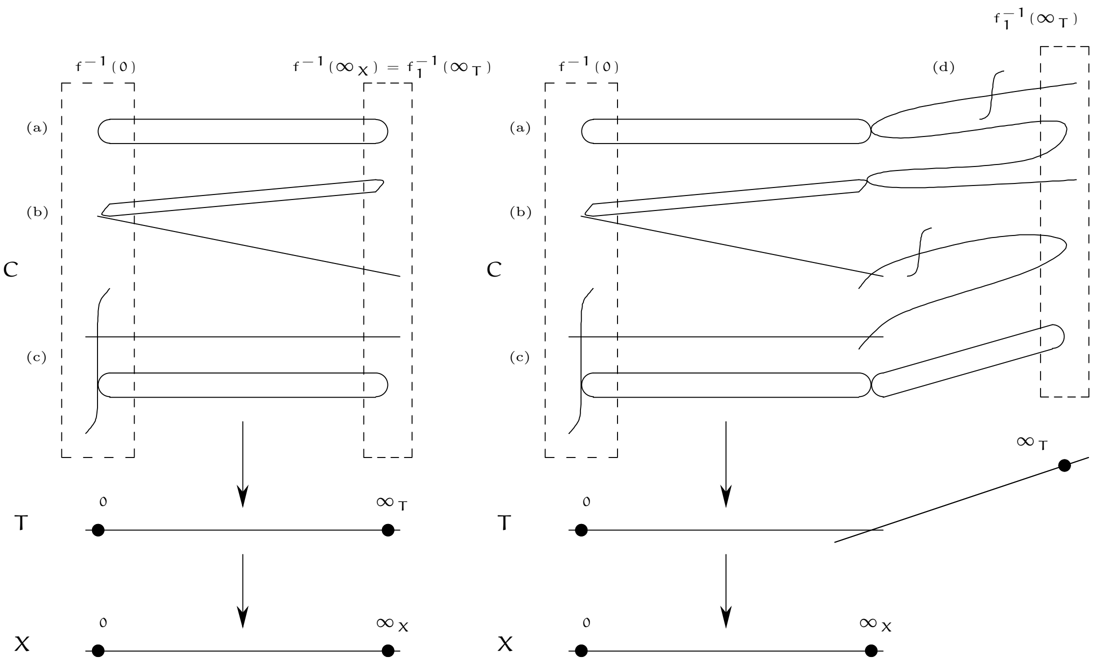

FIGURE 11. Two examples of torus-fixed stable relative maps to $(\mathbb{P}^1, \infty)$

so the torus acts with weight 1 on the tangent space at 0 and -1 on the tangent space at ∞. (The *weight* is the one-dimensional representation, or equivalently, the character.) This torus action induces an obvious torus action on $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)^\bullet$ (and $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)$).

We first determine the torus-fixed points of this action. Suppose $C \to T \to X$ is such a fixed map. A picture of two fixed maps showing "typical" behavior is given in Figure 11. The first has "nothing happening above ∞_X", and the second has some "sprouting" of T_i's.

The map $C \to X$ must necessarily be a covering space away from the points 0 and ∞ of $X = \mathbb{P}^1$.

Exercise. Using the Riemann-Hurwitz formula, show that a surjective map $C' \to \mathbb{P}^1$ from an irreducible curve, unbranched away from 0 and ∞, must be of the form $\mathbb{P}^1 \to \mathbb{P}^1$, $[x; y] \mapsto [x^a; y^a]$ for some a.

5.5. Hence the components dominating X must be a union of "trivial covers" of this sort.

We now focus our attention on the preimage of 0. Any sort of (stable) behavior above 0 is allowed. For example, the curve could be smooth and branched there (Figure 11(a)); or two of the trivial covers could meet in a node (Figure 11(b)); or there could be a contracted component of C, intersecting various trivial components at nodes (Figure 11(c)). (Because the "relative" part of the picture is at ∞, this discussion is the same as the discussion for ordinary stable maps, as discussed in [GrP].)

Finally, we consider the preimage of ∞_X. Possibly "nothing happens over ∞", i.e. the target has not sprouted a tree ($l = 0$ in our definition of stable relative maps at the start of §5), and the preimage of ∞ consists just of n smooth points; this is the first example in Figure 11. Otherwise, there is some "sprouting" of the target, and something "nontrivial" happens above each sprouted component T_i ($i > 0$), as in Figure 11(d).

5.6. At this point, you should draw some pictures, and convince yourself of the following important fact: the connected components of the fixed locus correspond to certain discrete data. In particular, each connected component can be interpreted as a product of three sorts of moduli spaces:

(A) moduli spaces of pointed curves (corresponding to Figure 11(c))

(B) (for those fixed loci where "something happens above ∞_X, i.e. Figure 11(d)), a moduli space of maps parametrizing the behaviour there. This moduli space is a variant of the space of stable relative maps, where there is no "rigidifying" map to X. We denote such a moduli space by $\overline{\mathcal{M}}_{g,\alpha,\beta}(\mathbb{P}^1, d)_\sim$; its theory (of deformations and obstructions and virtual fundamental classes) is essentially the same as that for $\overline{\mathcal{M}}_{g,\alpha,\beta}(\mathbb{P}^1, d)$. The virtual dimension of $\overline{\mathcal{M}}_{g,\alpha,\beta}(\mathbb{P}^1, d)_\sim$ is one less than that of $\overline{\mathcal{M}}_{g,\alpha,\beta}(\mathbb{P}^1, d)$.

(C) If $\alpha_1 + \cdots + \alpha_n = d$ is the partition corresponding to the "trivial covers" of T_0, these stable relative maps have automorphisms $\mathbb{Z}_{\alpha_1} \times \cdots \times \mathbb{Z}_{\alpha_n}$ corresponding to automorphisms of these trivial covers (i.e. if one trivial cover is of the form $[x; y] \mapsto [x^{\alpha_1}; y^{\alpha_1}]$, and ζ_{α_1} is a α_1th root of unity, then $[x; y] \mapsto [\zeta_{\alpha_1} x; y]$ induces an automorphism of the map). In the language of stacks, we can include a factor of $B\mathbb{Z}_{\alpha_1} \times \cdots \times B\mathbb{Z}_{\alpha_n}$; but the reader may prefer to simply divide the virtual fundamental class by $\prod \alpha_i$ instead.

Each of these spaces has a natural virtual fundamental class: the first sort has its usual fundamental class, and the second has its intrinsic virtual fundamental class.

The *relative virtual localization* formula states that

$$[\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)]^{\mathrm{virt}} = \sum_{\text{fixed}} [Y_i]^{\mathrm{virt}} / e(N_i^{\mathrm{virt}}),$$

in the equivariant homology of $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)$ (cf. (12)), with suitable terms inverted, where the virtual fundamental classes of the fixed loci are as just described, and the "virtual normal bundle" will be defined now.

Fix attention now to a fixed component Y_i. The virtual normal bundle is a class in equivariant K-theory. The term $1/e(N_i^{\mathrm{virt}})$ can be interpreted as the product of several factors, each "associated" to a part of the picture in Figure 11. We now describe these contributions. The reader is advised to not worry too much about the precise formulas; the most important thing is to get a sense of the shape of the formula upon a first exposure to these ideas. Let t be the generator of the equivariant cohomology of a point (i.e. $H_T^*(\mathrm{pt}) = \mathbb{Z}[t]$).

1. For each irreducible component dominating T_0 (i.e. each trivial cover) of degree α_i, we have a contribution of $\frac{\alpha_i^{\alpha_i}}{\alpha_i! t^{\alpha_i}}$.

2. For each contracted curve above 0 (Figure 11(c)) of genus g', we have a contribution of $(t^{g'}-\lambda_1 t^{g'-1}+\cdots+(-1)^{g'}\lambda_{g'})/t$. (This contribution is on the factor $\overline{\mathcal{M}}_{g',n}$ corresponding to the contracted curve.)

3. For each point where a trivial component of degree α_i meets a contracted curve above 0 at a point j, we have a contribution of $t/(t/\alpha_i - \psi_j)$. here, ψ_j is a class on the moduli space $\overline{\mathcal{M}}_{g',n}$ corresponding to the contracted component.

4. For each node above 0 (Figure 11(b)) joining trivial covers of degrees α_i and α_j, we have a contribution of $1/(t/\alpha_1 + t/\alpha_2)$.

5. For each smooth point above 0 (Figure 11(a)) on a trivial cover of degree α_i, we have a contribution of t/α_i.

At this point, if you squint and ignore the t's, you can almost see the ELSV formula (8) taking shape.

6. If there is a component over ∞_X, then we have a contribution of $1/(-t-\psi)$, where ψ is the first Chern class of the line bundle corresponding to the cotangent space of T_1 at the point where it meets T_0.

These six contributions look (and are!) complicated. But this formula can be judiciously used to give some powerful results, surprisingly cheaply. We now describe some of these.

6. Applications of relative virtual localization

6.1. Example 1: proof of the ELSV formula (8).

As a first example, we prove the ELSV formula (8). (This formula follows [GrV1], using the simplification in the last section of [GrV1] provided by the existence of Jun Li's description of the moduli space of stable relative maps.) The ELSV formula counts branched covers with specified branching over ∞ corresponding to $\alpha \vdash d$, and other fixed simple branched points. Hence we will consider $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)$.

We next need to impose other fixed branch points. There is a natural Gromov-Witten-theoretic approach involving using descendant invariants, but this turns out to be the wrong thing to do. Instead, we use a beautiful construction of Fantechi and Pandharipande [FanP]. Given any map from a nodal curve to $\mathbb{P}^1$, we can define a *branch divisor* on the target. When the source curve is smooth, the definition is natural (and old): above a point p corresponding to a partition $\beta \vdash d$, the branch divisor contains p with multiplicity $\sum(\beta_i - 1)$. It is not hard to figure out how extend this to the case where the source curve is not smooth above p.

Exercise. Figure out what this extension should be. (Do this so that the Riemann-Hurwitz formula remains true.)

Thus we have a map of sets $\overline{\mathcal{M}}_g(\mathbb{P}^1, d) \to \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1$. In the case of stable relative maps, we have a map of sets $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d) \to \mathrm{Sym}^{2d+2g-2}\mathbb{P}^1$. As each such stable relative map will have branching of at least $\sum(\alpha_i - 1)$ above ∞, we can subtract this fixed branch divisor to get a map of sets

$$\mathrm{br} : \overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d) \to \mathrm{Sym}^r \mathbb{P}^1 \tag{13}$$

where $r = 2d + 2g - 2 - \sum(\alpha_i - 1) = 2g - 2 + d + n$ (cf. (10)).

The important technical result proved by Fantechi and Pandharipande is the following.

6.2. *Theorem (Fantechi-Pandharipande* [FanP]*). — There is a natural map of stacks* br *as in* (13).

We call such a map a *(Fantechi-Pandharipande) branch morphism*. This morphism respects the torus action.

One can now readily verify several facts. If the branch divisor does not contain $p \neq \infty$ in $\mathbb{P}^1$, then the corresponding map $C \to \mathbb{P}^1$ is unbranched (i.e. a covering space, or étale) above p. If the branch divisor contains $p \neq \infty$ with multiplicity 1, then the corresponding map is simply branched above p. (Recall that this means that the preimage of p consists of smooth points, and the branching corresponds to the partition $2 + 1 + \cdots + 1$.) If the branch divisor does not contain ∞, i.e. there is no additional branching above ∞_X beyond that required by the definition of stable relative map, then the preimage of ∞_X consists precisely of the n smooth points q_i. In other words, there is no "sprouting" of T_i, i.e. $T \cong \mathbb{P}^1$. Hence if $p_1+\cdots+p_r$ is a general point of $\mathrm{Sym}^r\mathbb{P}^1$, then $\mathrm{br}^{-1}(p_1+\cdots+p_r) \subset \overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)$ is a finite set of cardinality equal to the Hurwitz number H^g_α. This is true despite the fact that $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)$ is horribly non-equidimensional — the preimage of a general point of $\mathrm{Sym}^r\mathbb{P}^1$ will be contained in $\mathcal{M}_{g,\alpha}(\mathbb{P}^1, d)$, and will not meet any other nasty components!

By turning this set-theoretic argument into something more stack-theoretic and precise, we have that

$$H^g_\alpha = \deg \mathrm{br}^{-1}(\mathrm{pt}) \cap [\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)]^{\mathrm{virt}}. \tag{14}$$

(For distracting unimportant reasons, the previous paragraph's discussion is slightly incorrect in the case where $H^g_\alpha = 1/2$, but (14) is true.)

We can now calculate the right side of (14) using localization. In order to do this, we need to interpret it equivariantly, which involves choosing an equivariant lift of br^{-1} of a point in $\mathrm{Sym}^r\mathbb{P}^1 \cong \mathbb{P}^r$. We do this by choosing our point in $\mathrm{Sym}^r\mathbb{P}^1$ to be the point 0 with multiplicity r. Thus all the branching (aside from that forced to be at ∞) must be at 0. The normal bundle to this point of $\mathbb{P}^r$ is $r!t^r$. Thus when we apply localization, a miracle happens. The only fixed loci we consider are those where there is no extra branching over ∞ (see the first picture in Figure 11). However, the source curve is smooth, so there is in fact only one connected component of the fixed locus to consider, which is shown in Figure 12. The moduli space in this case is $\overline{\mathcal{M}}_{g,n}$, which we take with multiplicity $1/\prod \alpha_i$ (cf. §5.6(C)). Hence the Hurwitz number is the intersection on this moduli space of the contributions to the virtual normal bundle outlined above.

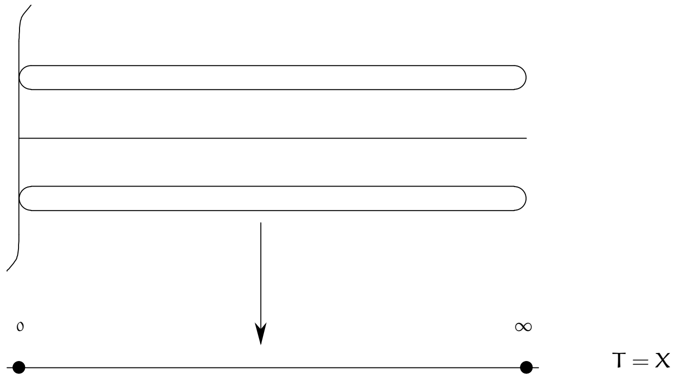

FIGURE 12. The only fixed locus contributing to our calculation of the Hurwitz number

Exercise. Verify that the contributions from *1*, *2*, and *3* above, on the moduli space $\overline{\mathcal{M}}_{g,n}$, give the ELSV formula (8).

6.3. Example 2: Proof of Theorem $\star$ (Theorem 4.1).

In Example 1 (§6.1), we found an equality of numbers. Here we will use relative virtual localization to get equality of cohomology or Chow classes.

Fix g and n. We are interested in dimension j (tautological) classes on $\overline{\mathcal{M}}_{g,n}$. In particular, we wish to show that any such *tautological* class can be deformed into one supported on the locus corresponding to curves with at least $2g-2+n-j$ genus 0 components. (This is just a restatement of Theorem $\star$.) Call such a dimension j class *good*. Using the definition of the tautological ring in terms of ψ-classes, it suffices to show that monomials in ψ-classes of dimension j (hence degree=codimension $3g-3+n-j$) are good.

Here is one natural way of getting dimension j classes. Take any partition $\alpha_1+\cdots+\alpha_n = d$. Let $r = 2g-2+n+d$ be the virtual dimension of $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)$ (i.e. the dimension of the virtual fundamental class, and the dimension of $\mathcal{M}_{g,\alpha}(\mathbb{P}^1, d)$), and suppose $r > j$. Define the *Hurwitz class* $\mathbb{H}_j^{g,\alpha}$ by

$$\mathbb{H}_j^{g,\alpha} := \pi_* \left((\cap_{i=1}^{r-j} br^{-1}(p_i)) \cap [\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)]^{\mathrm{virt}} \right) \in A_j(\overline{\mathcal{M}}_{g,n})$$

where π is the moduli map $\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d) \to \overline{\mathcal{M}}_{g,n}$ (and the n points are the preimages of ∞), and $p_1, \ldots, p_{r-j}$ are generally chosen points on $\mathbb{P}^1$. We think of this Hurwitz class informally as follows: consider branched covers with specified branching over ∞. Such covers (and their generalization, stable relative maps) form a space of (virtual) dimension r. Fix all but j branch points, hence giving a class of dimension j. Push this class to the moduli space $\overline{\mathcal{M}}_{g,n}$.

We get at this in two ways, by deformation and by localization.

1. Deformation. (We will implicitly use Li's degeneration formula here.) Deform the target $\mathbb{P}^1$ into a chain of $r-j$ $\mathbb{P}^1$'s, each with one of the fixed branch points p_i. Then you can (and should) check that the stabilized source curve has lots of rational components, essentially as many as stated in Theorem $\star$. (For example, imagine that $r \gg 0$. Then the j "roving" branch points can lie on only a small number of the $r-j$ components of the degenerated target. Suppose $\mathbb{P}^1$ is any other component of the target, where 0 and ∞ correspond to where it meets the previous and next component in the chain. Then the cover restricted to this $\mathbb{P}^1$ can have arbitrary branching over 0 and ∞, and only one other branch point: simple branching above the p_i lying on it. This forces the cover to be a number of trivial covers, plus one other cover $C \to \mathbb{P}^1$, where C is simply branched at p_i, and has one point above 0 and two points above ∞, or vice versa, forcing C to be genus 0, with three node-branches. This analysis will leave you slightly short. The remaining rational curves are found through a clever idea of Ionel, who seemingly conjures a rational curve out of nowhere.) Thus any dimension j Hurwitz class is *good*, i.e. satisfies the conclusion of Theorem $\star$.

2. Localization. We next use localization to express tautological classes in terms of Hurwitz classes. In the same way as for the ELSV formula, we choose an equivariant lifting of $\cap_{i=1}^{r-j} br^{-1}(p_i)$, corresponding to requiring all the p_i to go to 0. (Unlike the ELSV case, there are still j branch points that could go to either 0 or ∞.)

We now consider what fixed components can arise.

We have one "main" component that is similar to the ELSV case, where all the j "roving" branch points go to 0 (Figure 12). Any other component will be nontrivial over ∞. One can readily inductively show that these other components are *good*, i.e. satisfy the conclusion of Theorem $\star$. (The argument is by looking at the contribution from such a fixed locus. The part contained in $f^{-1}(\infty_X)$ is essentially a Hurwitz class, which we have shown is *good*. The part contained in $f^{-1}(0)$ corresponds to tautological classes on moduli spaces of curves with smaller $2g-2+n$, which can be inductively assumed to be *good*.)

Thus we have shown that the contribution of the "main" component is *good*. But this contribution is straightforward to contribute: it is (up to a multiple) the dimension j component of

$$\frac{1-\lambda_1+\cdots+(-1)^g\lambda_g}{(1-\alpha_1\psi_1)\cdots(1-\alpha_n\psi_n)}$$

(compare this to the ELSV formula (8)). By expanding this out, we find a polynomial in the α_i of degree $3g-3+n-j$ (cf. (9) for a similar argument earlier). We then apply the same trick as when we computed top intersections of ψ-classes using Hurwitz numbers in §3.20: we can recover the coefficients in this polynomial by "plugging in enough values". In other words, $\psi_1^{a_1}\cdots\psi_n^{a_1}$ may be obtained (modulo *good* classes) as a linear combination of Hurwitz classes. As Hurwitz classes are themselves *good*, we have shown that the monomial $\psi_1^{a_1}\cdots\psi_n^{a_1}$ is also *good*, completing the argument.

7. Towards Faber's intersection number conjecture 3.23 via relative virtual localization

We can use the methods of the proof of Theorem $\star$ to combinatorially describe the top intersections in the tautological ring. Using this, one can prove the "vanishing" or "socle" portion of the Faber-type conjecture for curves with rational tails (and hence for $\mathcal{M}_g$), and prove Faber's intersection number conjecture for up to three points. Details will be given in [GJV3]; here we will just discuss the geometry involved.

The idea is as follows. We are interested in the Chow ring of $\mathcal{M}^{rt}_{g,n}$, so we will work on compact moduli spaces, but discard any classes that vanish on the locus of curves with rational tails. We make a series of short geometric remarks.

First, note that $R_{2g-1}(\mathcal{M}^{rt}_{g,n}) \to R_{2g-1}(\mathcal{M}_{g,1})$ is an isomorphism, and $R_{2g-1}(\mathcal{M}_{g,1}) \to R_{2g-1}(\mathcal{M}_g)$ is a surjection. The latter is immediate from our definition. The argument for the former is for example [GrV3, Prop. 5.8], and can be taken as an exercise for the reader using Theorem $\star$. Faber showed [Fab1, Thm. 2] that $R_{2g-1}(\mathcal{M}_g)$ is non-trivial, so if we can show that $R_{2g-1}(\mathcal{M}_{g,1})$ is generated by a single element, then we will have proved that $R_{2g-1}(\mathcal{M}^{rt}_{g,n}) \cong \mathbb{Q}$ for all $n \geq 0$.

7.1. An extension of that argument using Theorem $\star$ shows that if we have a Hurwitz class of dimension less than $2g-1$ (i.e. with fewer than $2g-1$ "moving branch points"), then the class is 0 in $A_*(\mathcal{M}^{rt}_{g,n})$.

In order to get a hold of $R_{2g-1}(\mathcal{M}^{rt}_{g,n})$, we will again use branched covers. Before getting into the Gromov-Witten theory, we make a series of remarks, that may be verified by the reader, using only the Riemann-Hurwitz formula (2).

7.2. First, suppose we have a map $C \to \mathbb{P}^1$ from a nodal (possibly disconnected) curve, unbranched away from 0 and ∞. Then it is a union of trivial covers (in the sense of §5.5).

7.3. Second, suppose we have a map from a nodal curve C to $\mathbb{P}^1$, with no branching away from 0 and ∞ except for simple branching over 1, and nonsingular over 0 and ∞. Then it is a union of trivial covers, plus one more component, that is genus 0, completely branched over one of $\{0, \infty\}$, and with two preimages over the other. More generally, suppose we have a map from some curve C to a chain of $\mathbb{P}^1$'s, satisfying the kissing condition, unbranched except for two smooth points 0 and ∞ on the ends of the chain, and simple branching over another point 1. Then the map is the union of a number of trivial covers glued together, plus one other cover $\mathbb{P}^1 \to \mathbb{P}^1$ of the component containing 1, of the sort described in the previous sentence.

7.4. Third, if we have a map from a nodal curve C to $\mathbb{P}^1$, with total branching away from 0 and ∞ of degree less than $2g$, and nonsingular over 0 and ∞, then C has no component of geometric genus g. In the same situation, if the total branching away from 0 and ∞ is exactly $2g$, and C has a component of geometric genus g, then the cover is a disjoint union of trivial covers, and one connected curve C' of arithmetic genus g, where the map $C' \to \mathbb{P}^1$ is contracted to 1 or completely branched over 0 and ∞.

More generally, if we have a map from a curve C to a chain of $\mathbb{P}^1$'s satisfying the kissing condition, with 0 and ∞ points on either ends of the chain, with total branching away less than $2g$ away from 0, ∞, and the nodes, then C has no component of geometric genus g. In the same situation, if the total branching away from 0, ∞, and the nodes is precisely $2g$, then the map is the union of a number of trivial covers glued together, plus one other cover of the sort described in the previous paragraph.

7.5. The following fact is trickier. Let $\mathbb{G}_{g,d}$ be the image in $A_{2g-1}(\mathcal{M}_{g,1})$ of $\mathrm{br}^{-1}(1) \cap [\overline{\mathcal{M}}_{g,(d),(d)}(\mathbb{P}^1, d)]^{\mathrm{virt}}$ (where the point $p \in C$ corresponding to $[(C,p)] \in \mathcal{M}_{g,1}$ is the preimage of ∞). Then $\mathbb{G}_{g,d} = d^{2g}\mathbb{G}_{g,1}$. (We omit the proof, but the main idea behind this is the Fourier-Mukai fact [Lo, Lemma 2.10].)

Define the Faber-Hurwitz class $\mathbb{F}^{g,\alpha}$ as the image in $A_{2g-1}(\mathcal{M}^{\mathrm{rt}}_{g,n})$ of

$$\cap_{i=1}^{r-(2g-1)} \mathrm{br}^{-1}(p_i) \cap [\overline{\mathcal{M}}_{g,\alpha}(\mathbb{P}^1, d)]^{\mathrm{virt}}$$

where the p_i are general points of $\mathbb{P}^1$. (This is the image of the Hurwitz class $\mathbb{H}^{g,\alpha}_{2g-1}$ in $\mathcal{M}^{\mathrm{rt}}_{g,n}$.)

As with the proof of Theorem $\star$, we get at this class inductively using degeneration, and connect it to intersections of ψ-classes using localization.

7.6. Degeneration.

Break the target into two pieces $\mathbb{P}^1 \rightsquigarrow \mathbb{P}^1 \cup \mathbb{P}^1$, where ∞ and one p_i are on the "right" piece, and the remaining p_i's are on the "left" piece. The Faber-Hurwitz class breaks into various pieces; we enumerate the possibilities. We are interested only in components where there is a nonsingular genus g curve on one side. We have two cases, depending on whether this curve maps to the "left" or the "right" $\mathbb{P}^1$.

7.7. If it maps to the left component, then all $2g-1$ "moving" branch points must also map to the left component in order to get a non-zero contribution in $A_*(\mathcal{M}^{\mathrm{rt}}_{g,n})$, by Remark 7.4. Thus by Remark 7.3, the cover on the right is of a particular sort, and the cover on the left is another Faber-Hurwitz class, where one of the branch points over ∞ has been replaced two, or where two of the branch points are replaced by one.

7.8. If the genus g curve maps to the right component, then all $2g-1$ "moving" branch points must map to the right component, and by Remark 7.4 our contribution is a certain multiple of $\mathbb{G}_{g,d}$, which by Remark 7.5 is a certain multiple of $\mathbb{G}_{g,1}$. The contribution from the left is the genus 0 Hurwitz number H^0_α, for which Hurwitz gives us an attractive formula (7).

Unwinding this gives the recursion
(15)

$$\mathbb{F}^{g,\alpha} = \sum_{\substack{i+j=\alpha_k \\ d'+d''=d}} ij H^0_{\alpha'} \mathbb{F}^{g,\alpha''} \binom{d+l(\alpha)-2}{d'+l(\alpha')-2,\, d''+l(\alpha'')-1} + \sum_{i,j} (\alpha_i+\alpha_j)\, \mathbb{F}^{g,\alpha'} + \sum_{i=1}^{l(\alpha)} \alpha_i^{2g+1} H^0_\alpha \mathbb{G}_{g,1}.$$

In this formula, the contributions from paragraph §7.7 are in the first two terms on the right side of the equation, and the contributions from §7.8 are in the last. The second term on the right corresponds to where two parts α_i and α_j of α are "joined" by the nontrivial cover of the right $\mathbb{P}^1$ to yield a new partition where α_i and α_j are replaced by $\alpha_i + \alpha_j$. The first term on the right corresponds to where one part α_k of α is "cut" into two pieces i and j, forcing the curve covering the left $\mathbb{P}^1$ to break into two pieces, one of genus 0 (corresponding to partition α') and one of genus g (corresponding to α''). The binomial coefficient corresponds to the fact that the $d + l(\alpha) - 2$ fixed branch points $p_1, p_2, \dots$ on the left component must be split between these two covers.

The base case is $\mathbb{F}^{g,(1)} = \mathbb{G}_{g,1}$. Hence we have shown that $\mathbb{F}^{g,\alpha}$ is always a multiple of $\mathbb{G}_{g,1}$, and the theory of cut-and-join type equations (developed notably by Goulden and Jackson) can be applied to solve for $\mathbb{F}^{g,\alpha}$ (in generating function form) quite explicitly.

7.9. Localization.

We now get at the Faber-Hurwitz class by localizing. As with the proof of Theorem $\star$, we choose a linearization on $br^{-1}(p_i)$ that corresponds to requiring all the p_i to move to 0. We now describe the fixed loci that contribute. We won't worry about the precise contribution of each fixed locus; the important thing is to see the shape of the formula.

First note that as we have only $2g-1$ moving branch points, in any fixed locus in the "rational-tails" locus, our genus g component cannot map to ∞, and thus must be contracted to 0. The fixed locus can certainly have genus 0 components mapping to sprouted T_i over ∞, as well as genus 0 components contracted to 0.

We now look at the contribution of this fixed locus, via the relative virtual localization formula. We will get a sum of classes glued together from various moduli spaces appearing in the description of the fixed locus (cf. §5.6). Say the contracted genus g curve meets m trivial covers, of degree $\beta_1, \dots, \beta_m$ respectively. Then the contribution from this component will be some summand of

$$\frac{1 - \lambda_1 + \cdots + (-1)^g \lambda_g}{(1-\beta_1\psi_1)\cdots(1-\beta_m\psi_m)}$$

where ψ_i are the ψ-classes on $\overline{\mathcal{M}}_{g,m}$. Thus the contribution from this component is visibly tautological, and by Remark 7.1 the contribution will be zero if the dimension of the class is less than $2g-1$. As the total contribution of this fixed locus is $2g-1$, any non-zero contribution must correspond to a dimension $2g-1$ tautological class on $\overline{\mathcal{M}}_{g,m}$ glued to a dimension 0 class on the other moduli spaces appearing in this fixed locus. This can be readily computed; the genus 0 components contracted to 0 yield binomial coefficients, any components over ∞_X turn out to yield products of *genus 0 double Hurwitz numbers*, which count branched covers of $\mathbb{P}^1$ by a genus 0 curve, with specified branching α and β above two points, and the remaining branching fixed and simple.

Equipped with this localization formula, even without worrying about the specific combinatorics, we may show the following.

7.10. *Theorem. — For any n, and $\beta \vdash d$, $\pi_* \psi_1^{\beta_1} \cdots \psi_n^{\beta_n}$ is a multiple of $\mathbb{G}_{d,1}$, where π is the forgetful map to $\mathcal{M}_{g,1}$.*

We have thus fully shown the "vanishing" (or socle) part of Faber's conjecture for curves with rational tails. (This may certainly be shown by other means.) In particular, we have completed a proof of Looijenga's Theorem 4.5.

Proof. Call such a class an n-point class. We will show that such a class is a multiple of $G_{d,1}$ modulo m-point classes, where $m < n$; the result then follows by induction. As with the proof of Theorem $\star$, we consider $\mathbb{F}^{g,\alpha}$ as α varies over all partitions of length n. Each such Faber-Hurwitz class is a multiple of $\mathbb{G}_{d,1}$ by our degeneration analysis. By our localization analysis, all of the fixed points for $\mathbb{F}^{g,\alpha}$ yield m-point classes where $m < n$ except for one, corresponding to the picture in Figure 12. The contribution of this component is some known multiple of a polynomial in $\alpha_1, \ldots, \alpha_n$. The highest-degree coefficients of this polynomial are the n-point classes, the monomials in ψ-classes that we seek. By taking a suitable linear combination of values of the polynomial (i.e. Faber-Hurwitz classes, modulo m-point classes where $m < n$), we can obtain any coefficient, and in particular, the leading coefficients. □

A related observation is that we have now given an explicit combinatorial description of the monomials in ψ-classes, as a multiple of our generator $\mathbb{G}_{g,1}$. (In truth, we have not been careful in this exposition in describing all the combinatorial factors. See [GJV3] for a precise description.)

This combinatorialization can be made precise as follows. We create a generating function $\mathbb{F}$ for Faber-Hurwitz classes. The join-cut equation (15) allows us to solve for the generating function $\mathbb{F}$.

We make a second generating function $\mathbb{W}$ for the intersections $\pi_* \psi_1^{\beta_1} \cdots \psi_n^{\beta_n} \lambda_k \in R_{2g-1}(\mathcal{M}_{g,1})$ (where $\beta_1 + \cdots + \beta_n + k = g - 2$). Localization gives us a description of $\mathbb{F}$ in terms of $\mathbb{W}$ (and also the genus 0 double Hurwitz generating function). By inverting this relationship we can hope to solve relatively explicitly for $\mathbb{W}$, and hence prove Faber's intersection number conjecture. Because genus 0 double Hurwitz number $H^0_{\alpha,\beta}$ are only currently well-understood when one of the partition has at most 3 parts (see [GJV2]), this program is not yet complete. However, it indeed yields:

7.11. *Theorem* [GJV3]. — *Faber's intersection number conjecture is true for up to three points.*

One might reasonably hope that this will give an elegant proof of Faber's intersection number conjecture in full before long.

8. CONCLUSION

In the last fifteen years, there has been a surge of progress in understanding curves and their moduli using the techniques of Gromov-Witten theory. Many of these techniques have been outlined here.

Although this recent progress uses very modern machinery, it is part of an ancient story. Since the time of Riemann, algebraic curves have been studied by way of branched covers of $\mathbb{P}^1$. The techniques described here involve thinking about curves in the same way. Gromov-Witten theory gives the added insight that we should work with a "compactification" of the space of branched covers, the moduli space of stable (relative) maps. A priori we pay a steep price, by working with a moduli space that is bad in all possible ways (singular, reducible, not even equidimensional). But it is in some sense "virtually smooth", and its virtual fundamental class behaves very well, in particular with respect to degeneration and localization.

The approaches outlined here have one thing in common: in each case the key idea is direct and naive. Then one works to develop the necessary Gromov-Witten-theoretic tools to make the naive idea precise.

In conclusion, the story of using Gromov-Witten theory to understand curves, and to understand curves by examining how they map into other spaces (such as $\mathbb{P}^1$), is most certainly not over, and may just be beginning.

REFERENCES

[ArbC] E. Arbarello and M. Cornalba, *The Picard groups of the moduli spaces of curves*, Topology **26** (1987), 153–171.

[ArcL1] D. Arcara and Y.-P. Lee, *Tautological equations in genus 2 via invariance conjectures*, preprint 2005, math.AG/0502488.

[ArcL2] D. Arcara and Y.-P. Lee, *Tautological equations in* $\overline{M}_{3,1}$ *via invariance conjectures*, preprint 2005, math.AG/0503184.

[ArcL3] D. Arcara and Y.-P. Lee, *On independence of generators of the tautological rings*, preprint 2006, math.AG/0605488.

[ArcS] D. Arcara and F. Sato, *Recursive formula for* $\psi^g - l a_1 \psi^{g-1} + \cdots + (-1)^g \lambda_g$ *in* $\overline{\mathcal{M}}_{g,1}$, preprint 2006, math.AG/0605343.

[Arn] V. I. Arnol'd, *Topological classification of trigonometric polynomials and combinatorics of graphs with an equal number of vertices and edges*, Funct. Anal. and its Appl. **30** no. 1 (1996), 1–17.

[AB] M. Atiyah and R. Bott, *The moment map and equivariant cohomology*, Topology **23** (1984), 1–28.

[BP] P. Belorousski and R. Pandharipande, *A descendent relation in genus 2*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), no. 1, 171–191.

[C] K. Costello, *Higher-genus Gromov-Witten invariants as genus 0 invariants of symmetric products*, preprint 2003, math.AG/0303387.

[CK] D. Cox and S. Katz, *Mirror Symmetry and Algebraic Geometry*, Mathematical surveys and Monographs **68**, Amer. Math. Soc., Providence, RI, 1999.

[CT] M. Crescimanno and W. Taylor, *Large* N *phases of chiral* QCD_2, Nuclear Phys. B **437** (1995), 3–24.

[D] J. Dénes, *The representation of a permutation as the product of a minimal number of transpositions and its connection with the theory of graphs*, Publ. Math. Ins. Hungar. Acad. Sci. **4** (1959), 63–70.

[DVV] R. Dijkgraaf, H. Verlinde, and E. Verlinde, *Topological strings in* $d < 1$, Nuclear Phys. B **352** (1991), 59–86.

[E] D. Edidin, *Notes on the construction of the moduli space of curves*, in *Recent progress in intersection theory (Bologna, 1997)*, 85–113, Trends Math., Birkhäuser Boston, Boston, MA, 2000.

[ELSV1] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *On Hurwitz numbers and Hodge integrals*, C. R. Acad. Sci. Paris Sér. I Math. **328** (1999), 1175–1180.

[ELSV2] T. Ekedahl, S. Lando, M. Shapiro, and A. Vainshtein, *Hurwitz numbers and intersections on moduli spaces of curves*, Invent. Math. **146** (2001), 297–327.

[Fab1] C. Faber, *A conjectural description of the tautological ring of the moduli space of curves*, in *Moduli of Curves and Abelian Varieties*, 109–129, Aspects Math., **E33**, Vieweg, Braunschweig, 1999.

[Fab2] C. Faber, MAPLE program for computing Hodge integrals, personal communication. Available at `http://math.stanford.edu/~vakil/programs/`.

[Fab3] C. Faber, *Algorithms for computing intersection numbers on moduli spaces of curves, with an application to the class of the locus of Jacobians*, in *New trends in algebraic geometry (Warwick, 1996)*, 93–109, London Math. Soc. Lecture Note Ser., 264, Cambridge Univ. Press, Cambridge, 1999.

[Fab4] C. Faber, personal communication, January 8, 2006.

[FabP1] C. Faber and R. Pandharipande, *Logarithmic series and Hodge integrals in the tautological ring*, Michigan Math. J. (Fulton volume) **48** (2000), 215–252.

[FabP2] C. Faber and R. Pandharipande, *Hodge integrals, partition matrices, and the* λ_g *conjecture*, Ann. Math. **157** (2003), 97–124.

[FabP3] C. Faber and R. Pandharipande, *Relative maps and tautological classes*, J. Eur. Math. Soc. **7** (2005), no. 1, 13–49.

[Fan] B. Fantechi, *Stacks for everybody*, in *European Congress of Mathematics, Vol. I (Barcelona, 2000)*, 349–359, Progr. Math., **201**, Birkhäuser, Basel, 2001.

[FanP] B. Fantechi and R. Pandharipande, *Stable maps and branch divisors*, Compositio Math. **130** (2002), no. 3, 345–364.

[Ga] A. Gathmann, *Absolute and relative Gromov-Witten invariants of very ample hypersurfaces*, Duke Math. J. **115** (2002), 171–203.

[Ge1] E. Getzler, *Intersection theory on* $\overline{\mathcal{M}}_{1,4}$ *and elliptic Gromov-Witten invariants*, J. Amer. Math. Soc. **10** (1997), no. 4, 973–998.

[Ge2] E. Getzler, *Topological recursion relations in genus 2*, in *Integrable systems and algebraic geometry (Kobe/Kyoto, 1997)*, 73–106, World Sci. Publishing, River Edge, NJ, 1998.

[GeP] E. Getzler and R. Pandharipande, *Virasoro constraints and the Chern classes of the Hodge bundle*, Nuclear Phys. B **530** (1998), 701–714.

[GiaM] J. Giansiracusa and D. Maulik, *Topology and geometry of the moduli space of curves*, on-line collection of resources on moduli of curves and related subjects, `http://www.aimath.org/WWN/modspacecurves/`.

[Giv] A. Givental, *Gromov-Witten invariants and quantization of quadratic hamiltonians*, Mosc. Math. J. **1** (2001), no. 4, 551–568, 645.

[GJ1] I. P. Goulden and D. M. Jackson, *Transitive factorizations into transpositions and holomorphic mappings on the sphere*, Proc. Amer. Math. Soc. **125** (1997), 51–60.

[GJ2] I. P. Goulden and D. M. Jackson, *The number of ramified coverings of the sphere by the double torus, and a general form for higher genera*, J. Combin. Theory A **88** (1999) 259–275.

[GJV1] I. P. Goulden, D. M. Jackson, R. Vakil, *The Gromov-Witten potential of a point, Hurwitz numbers, and Hodge integrals*, Proc. London Math. Soc. (3) **83** (2001), 563–581.

[GJV2] I. P. Goulden, D. M. Jackson, R. Vakil, *Towards the geometry of double Hurwitz numbers*, Adv. Math. (Artin issue) **198** (2005), 43–92.

[GJV3] I. P. Goulden, D. M. Jackson, R. Vakil, *On Faber's intersection number conjecture on the moduli space of curves*, in preparation.

[GrP] T. Graber and R. Pandharipande, *Localization of virtual classes*, Invent. Math. **135** (1999), 487–518.

[GrV1] T. Graber and R. Vakil, *Hodge integrals and Hurwitz numbers via virtual localization*, Compositio Math. **135** (2003), no. 1, 25–36.

[GrV2] T. Graber and R. Vakil, *On the tautological ring of* $\overline{\mathcal{M}}_{g,n}$, in *Proceedings of the Seventh Gökova Geometry-Topology Conference 2000*, International Press, 2000.

[GrV3] T. Graber and R. Vakil, *Relative virtual localization and vanishing of tautological classes on moduli spaces of curves*, Duke Math. J. **130** (2005), no. 1, 1–37.

[HLo] R. Hain and E. Looijenga, *Mapping class groups and moduli spaces of curves*, Proc. Sympos. Pure Math. **62** Part 2, pp. 97–142, Amer. Math. Soc., Providence, RI, 1997.

[HZ] J. Harer and D. Zagier, *The Euler characteristic of the moduli space of curves*, Invent. Math. **85** (1986), no. 3, 457–485.

[HM] J. Harris and I. Morrison, *Moduli of Curves*, Graduate Texts in Mathematics **187**, Springer-Verlag, New York, 1998.

[HKKPTVVZ] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, and E. Zaslow, *Gromov-Witten Theory and Mirror Symmetry*, Clay Math. Inst., Amer. Math. Soc., 2002.

[H] A. Hurwitz, *Über Riemann'sche Flächen mit gegeben Verzweigungspunkten*, Math. Ann. **39** (1891), 1–60.

[I1] E. Ionel, *Topological recursive relations in* $H^{2g}(M_{g,n})$, Invent. Math. **148** (2002), no. 3, 627–658.

[I2] E. Ionel, *Relations in the tautological ring of* M_g, Duke Math. J. **129** (2005), no. 1, 157–186.

[IP1] E.-N. Ionel and T. H. Parker, *Relative Gromov-Witten invariants*, Ann. of Math. (2) **157** (2003), 45–96.

[IP2] E.-N. Ionel and T. H. Parker, *The symplectic sum formula for Gromov-Witten invariants*, Ann. of Math. (2), **159** (2004), 935–1025.

[Ke] S. Keel, *Intersection theory of moduli space of stable* n*-pointed curves of genus zero*, Trans. Amer. Math. Soc. **330** (1992), no. 2, 545–574.

[KiL] Y.-S. Kim and K. Liu, *A simple proof of Witten conjecture through localization*, preprint 2005, math.AG/0508384.

[KL] T. Kimura and X. Liu, *A genus-*3 *topological recursion relation*, preprint 2005, math.DG/0502457.

[Ko1] M. Kontsevich, *Intersection theory on the moduli space of curves and the matrix Airy function*, Comm. Math. Phys. **147** (1992), 1–23.

[Ko2] M. Kontsevich, *Enumeration of rational curves via torus actions*, in *the Moduli Space of Curves (Texel Island, 1994)*, R. Dijkgraaf, C. Faber and G. van der Geer, eds., Progr. Math. vol. 129, Birkhäuser, Boston, 1995, pp. 335–368.

[Kr] A. Kresch, *Cycle groups for Artin stacks*, Invent. Math. **138** (1999), no. 3, 495–536.

[Lee1] Y.-P. Lee, *Invariance of tautological equations I: Conjectures and applications*, preprint 2006, math.AG/0604318v2.

[Lee2] Y.-P. Lee, *Invariance of tautological equations II: Gromov-Witten theory*, preprint 2006, available at `http://www.math.utah.edu/~yplee/research/`.

[LeeP] Y.-P. Lee and R. Pandharipande, *Frobenius manifolds, Gromov-Witten theory, and Virasoro constraints*, book in preparation.

[Li1] J. Li, *Stable morphisms to singular schemes and relative stable morphisms*, J. Diff. Geom. **57** (2001), 509–578.

[Li2] J. Li, *A degeneration formula of GW-invariants*, J. Diff. Geom. **60** (2002), 199–293.

[LR] A.-M. Li and Y. Ruan, *Symplectic surgery and Gromov-Witten invariants of Calabi-Yau 3-folds*, Invent. Math. **145** (2001), 151–218.

[LLZ] C.-C. M. Liu, K. Liu, and J. Zhou, *A proof of a conjecture of Marino-Vafa on Hodge integrals*, J. Diff. Geom. **65** (2004), 289–340.

[Lo] E. Looijenga, *On the tautological ring of* $\mathcal{M}_g$, Invent. Math. **121** (1995), no. 2, 411–419.

[MT] I. Madsen and U. Tillmann, *The stable mapping class group and* $Q(\mathbb{CP}^\infty_+)$, Invent. Math. **145** (2001), no. 3, 509–544.

[MW] I. Madsen and M. Weiss, *The stable moduli space of Riemann surfaces: Mumford's conjecture*, preprint 2002, math.AT/0212321.

[Mi] M. Mirzakhani, *Weil-Petersson volumes and the Witten-Kontsevich formula*, preprint 2003.

[Mo1] S. Morita, *Generators for the tautological algebra of the moduli space of curves*, Topology **42** (2003), 787–819.

[Mo2] S. Morita, *Cohomological structure of the mapping class group and beyond*, preprint 2005, math.GT/0507308v1.

[Mu] D. Mumford, *Toward an enumerative geometry of the moduli space of curves*, in *Arithmetic and Geometry*, Vol. II, M. Artin and J. Tate ed., 271–328, Prog. Math. **36**, Birk. Boston, Boston, MA, 1983.

[OP] A. Okounkov and R. Pandharipande, *Gromov-Witten theory, Hurwitz numbers, and matrix models, I*, math.AG/0101147.

[P] R. Pandharipande, *Three questions in Gromov-Witten theory*, in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, 503–512, Higher Ed. Press, Beijing, 2002.

[R] B. Riemann, *Theorie der Abel'schen Funktionen*, J. Reine angew. Math. **54** (1857), 115–155.

[T] U. Tillmann, *Strings and the stable cohomology of mapping class groups*, in *Proceedings of the International Congress of Mathematicians, Vol. II (Beijing, 2002)*, 447–456, Higher Ed. Press, Beijing, 2002.

[V1] R. Vakil, *Genus* 0 *and* 1 *Hurwitz numbers: Recursions, formulas, and graph-theoretic interpretations*, Trans. Amer. Math. Soc. **353** (2001), 4025–4038.

[V2] R. Vakil, *The moduli space of curves and its tautological ring*, Notices of the Amer. Math. Soc. (feature article), vol. 50, no. 6, June/July 2003, p. 647–658.

[Vi] A. Vistoli, *Intersection theory on algebraic stacks and on their moduli spaces*, Invent. Math. **97** (1989), 613–670.

[W] E. Witten, *Two dimensional gravity and intersection theory on moduli space*, Surveys in Diff. Geom. **1** (1991), 243–310.

DEPT. OF MATHEMATICS, STANFORD UNIVERSITY, STANFORD CA 94305–2125

E-mail address: `vakil@math.stanford.edu`