A NONPROJECTIVE THREEFOLD

A. KNUTSON, AS TOLD TO R. VAKIL

(I make no claim of originality or non-originality on Allen's behalf.)

I'll present his example as a sequence of exercises.

Consider $\mathbb{P}^3$ with four non-coplanar points $p_1, \ldots, p_4$. Let X be the blow-up of $\mathbb{P}^3$ at the four points, with exceptional divisors $E_1, \ldots, E_4$. Let H be the divisor class that is the pullback of the hyperplane section.

1. Check that the divisor class $D = 2H - E_1 - E_2 - E_3 - E_4$ on X is basepoint free. (*Hint:* given any point, explicitly describe a divisor in the class not containing the point.) Let its image be Y.

2. Let $l_{ij} \subset X$ be the proper transform of the line joining p_i and p_j. Show that D contracts the six lines l_{ij}, and preserves the rest of X. (*Hint:* Show that the linear system separates enough points and tangent vectors.) In fact, these are small contractions; the resulting singularity is analytically equivalent to $x^2 + y^2 + w^2 + z^2 = 0$ (first studied by Atiyah?), but we won't need that fact.

3. These six contractions can be done independently. Let Y' be the partial desingularization of Y that corresponds to contracting all l_{ij} on X except l_{12}. Show that Y' is not projective. (*Hint:* If Y' is projective, then let the pullback of an ample divisor to X be $D' = aH - b_1E_1 - b_2E_2 - b_3E_3 - b_4E_4$. Then (by the projection formula) we have $D' \cdot l_{ij} = 0$ except $D' \cdot l_{12} > 0$.)

The example has a nice toric description as well, which is how he described it to me. The polytope corresponding to $\mathbb{P}^3$ is a tetrahedron (in a lattice). Blowing up the four points corresponds to slicing off the four vertices (yielding four new triangular faces). Blowing down the six lines corresponds to slicing off enough that the four new triangular faces mutually meet (i.e. the six edges are cut down to nothing). The combinatorial polytope corresponding to this example is one where five of the six edges are cut down to nothing, but the last is not. This is not achievable as a rational polytope. I don't know how to make this precise.

4. Make this precise.

E-mail address: `vakil@math.stanford.edu`

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