DEFORMATION THEORY WORKSHOP: SOME NOTES FROM JASON STARR'S TALK

ROUGH NOTES BY RAVI VAKIL

Here is Jason Starr's introduction to obstruction theory. (I realized this obstruction theory discussion would fit well with the lectures series some way into his talk, so I'm just typing up his discussion of obstruction theory; I didn't get a chance to type up his Mori discussion.)

Obstruction theory.

Let R be a local complete Noetherian ring, with algebraically closed residue field κ.

Let $\mathcal{C}_R$ be the category of local Artin R-algebras with residue field κ.

Infinitesimal extension $A' \xrightarrow{q} A$ such that $\ker(q) = N$ with $\mathfrak{m}_A N = 0$.

Thus N is a finite-dimensional κ-vector space.

$$\begin{array}{cccccccccc} \sigma: & 0 & \longrightarrow & N & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\ \downarrow u & & & \downarrow u_N & & \downarrow u_{A'} & & \downarrow u_A & & \\ \tilde{\sigma}: & 0 & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{A}' & \longrightarrow & \tilde{A} & \longrightarrow & 0 \end{array}$$

Let $F : \mathcal{C}_R \to$ **Sets** be a functor with $F(\kappa) = \{\bullet\}$.

Define a deformation situation $(\Sigma, x \in F(A))$.

An **obstruction theory** (O, ω) is the following.

- O is a finite-dimensional κ-vector space ($N \mapsto N \otimes_\kappa O$).
- ω is a rule $(\Sigma, w) \mapsto \omega_{\Sigma,x} \in N \otimes_\kappa O$:

(i) that is suitably natural, i.e. for all $u : (\Sigma, x) \to (\tilde{\Sigma}, \tilde{x})$, the image of $\omega_{\Sigma,x}$ under $N \otimes_\kappa O \to \tilde{N} \otimes_\kappa O$ is $\omega_{\tilde{\Sigma},\tilde{x}}$.

(ii) $\omega_{\Sigma,x}$ equals 0 iff x is the image of an element $x' \in F(A')$.

Exercise: (i) implies the "if" part of (ii).

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Examples: (1) Suppose F is representable: $F = h_S$,

$$S = R[[x]]/I = R[[\widehat{x_1, \ldots, x_n}]]/\langle f_1, \ldots, f_s \rangle.$$

$I/I^2 \to \hat{\Omega}_{R[[x]]/R} \otimes_{R[[x]]} S$

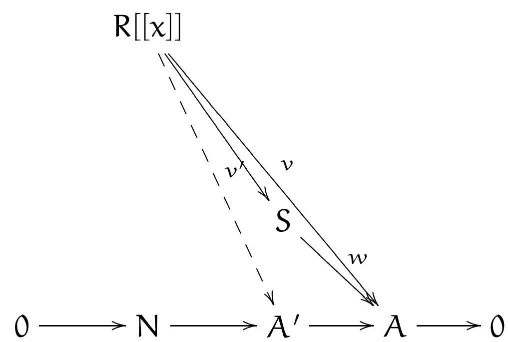
(Think: $\hat{\Omega}_{R[[x]]/R} = R[[x]]\{dx_1, \ldots, dx_r\}$.)

$\mathrm{Hom}_{R[[x]]}(\hat{\Omega}_{R[[x]]/R}, \kappa) \to \mathrm{Hom}_S(I/I^2, \kappa)$

$(\phi : dx_i \mapsto c_i) \mapsto (f_j \mapsto \sum_{i=1}^r \overline{\frac{\partial f_j}{\partial x_i}} c_i)$

We define O to be the cokernel of this map.

Here's why this works.



$\partial : dx_i \mapsto$ an element of N, giving $\partial : \hat{\Omega}_{R[[x]]/R} \to N$.

$\nu' + \partial$ factors through S iff $f_1, \ldots, f_s \mapsto 0$. $\nu' + \partial$ implies that we have an S-module homomorphism $I/I^2 \to N$, $f_j \mapsto (\nu' + \partial)(f_j)$.

The upshot of all this is that we have an element

$$\omega \in \mathrm{Hom}_S(I/I^2, N)/\mathrm{Hom}_{R[[x]]}(\hat{\Omega}, N) = (\mathrm{Hom}(I/I^2, \kappa)/\mathrm{Hom}(\hat{\Omega}, \kappa)) \otimes_\kappa N$$

is independent of the choice of ν'. This obstruction vanishes iff w extends to an R-algebra homomorphism $S \to A'$.

Example: Let C be a smooth projective connected curve over κ. Let $Z \subset C$ be an effective Cartier divisor. Let X be a smooth κ-scheme. Let $f_0 : C \to X$ be a κ-morphism. Denote $g := f_U|_Z : Z \to X$.

$R = \kappa$. $F : A \mapsto \{f_A : C \times_\kappa \mathrm{Spec}\, A \to X \times_k \mathrm{Spec}\, A \mid f_A$ is a $\mathrm{Spec}\, A$=morphism such that
(i) $f \equiv f_0$ modulo $\mathfrak{m}_A$, and
(ii) $f|_{Z \times \mathrm{Spec}\, A} = g \times \mathrm{Id}_{\mathrm{Spec}\, A}$. $\}$

To this I want to associate an obstruction theory. $O := H^1(C, f^*T_X \otimes \mathcal{I}_Z)$. I now need to tell you the obstructions. Given a deformation situation

$$0 \to N \to A' \to A \to 0,$$

and we have a map $f_A : C_A \to X_A$,

$$0 \to N \to A' \to A \to 0.$$

We'll do this affine open by affine open.

Let $\mathcal{U} \subset C$ be an affine open. There is an infinitesimal extension property (that we've discussed)

$$\begin{array}{ccc} & & X \\ & \overset{f_A|_{\mathcal{U}_A}}{\nearrow} & \uparrow \widetilde{f_{A,\mathcal{U}}} \\ \mathcal{U}_A & \hookrightarrow & \mathcal{U}_{A'} \end{array}$$

This map is not unique; but any other differs by a derivation $\Omega_X \to N \otimes f_* \mathcal{O}_{\mathcal{U}_A} \otimes \mathcal{I}_Z$.

If you unravel what is going on, it is not hard to check that they satisfy the cocycle condition. This has come up in Martin's lectures, so the discussion is omitted here.

Fact. Let $F = h_S$ be a prorepresentable functor on $\mathcal{C}_R$. Let $O_{can} := \operatorname{Hom}(I/I^2, \kappa)/\operatorname{Hom}(\widehat{\Omega}_{R[[x]]/R}, \kappa)$. Let O be *any* other obstruction theory. Then there exists a unique $\psi : O_{can} \to O$ such that every $\omega_{\Sigma,x} = \operatorname{im} \omega_{\Sigma,X,can}$ under ψ.

The idea under the proof is as follows (although you may see this next week). Hypothesis: your defining equations are quadratic or higher, i.e. $I \subset \mathfrak{m}^2_{R[[x]]}$.

$$\begin{array}{ccccccccc} 0 & \to & I/\mathfrak{m}I & \to & R[[x]]/(\mathfrak{m}_{R[[x]]}I + \mathfrak{m}^c) & \to & R[[x]]/(I + \mathfrak{m}_{R[[x]]}) = S/\mathfrak{m}_S^c & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & N & \to & A' & \to & A & \to & 0 \end{array}$$

$\mathfrak{m}^C_{A'} = 0$

But you don't just get this canonical map of obstruction theories; it is injective.

This is very handy.

$I/\mathfrak{m}_{R[[x]]}I$ is a free κ-vector space with basis the images a minimal set of generators for I.

$S = R[[x_1, \ldots, x_r]]/\langle f_1, \ldots, f_s \rangle$.

t_F has basis the duals of $dx_1, \ldots, dx_r$.

So $\dim_k t_F = r =$ the minimal number of generators.

Also, $\dim_k O \geq s =$ minimal number of relations.

Thus $\dim S \geq \dim R + r - s \geq \dim r + \dim t_F - \dim O$.

Thus if you have some first order deformation space and obstruction space given to you by nature, you get a bound on the dimension of the space.

I've only given you one example not including the representable functors, and let's return to that one now.

Example.

$$\begin{array}{ccc} C & \xrightarrow{f_u} & X \\ \uparrow & \nearrow_{g} & \\ Z & & \end{array}$$

and we want to "fill this in"

$$\begin{array}{ccccc} C & \longrightarrow & C_A & \longrightarrow & X \\ \uparrow & & \uparrow & \nearrow_{g} & \\ Z & \longrightarrow & Z_A & & \end{array}$$

Now $t_F = H^0(C, f^*T_X \otimes I)$ and $O = H^1(C, f^*T_X \otimes I)$, and the difference is the euler characteristic, which we get from Riemann-Roch.

You get more: if we have equality, then S is R-flat (and is a local complete intersection over R).

Moreover, if the dimension is positive, then $\dim_{[f_V]} \mathrm{Hom}(C, X; g : Z \to X) > 0$. And $\mathrm{Hom}(C, X; g : Z \to X) > 0$ is quasiprojective (once we fix the degree). So there is an affine curve $B \subset \mathrm{Hom}$ containing $[f_w]$. That means we have a map

$$\begin{array}{ccc} B \times C & \xrightarrow{f} & X \\ \uparrow & & \uparrow g \\ B \times Z & \xrightarrow{pr_Z} & Z \end{array}$$

If $\overline{B}$ is a projective completion, we can ask if we can extend f to get a map $\overline{B} \times C \to X$. Note that the thing on the left is a smooth surface. You can't necessarily extend the map, but you can after blowing up a finite number of times. If you do so a minimal number of times, then the final exceptional divisor can not map to a point on X (or else that was a redundant map).

We're now getting into the original theorem Jason was discussing (Mori's theorem), which I didn't type in, so I'll stop here.

E-mail address: vakil@math.stanford.edu