DEFORMATION THEORY WORKSHOP: OSSERMAN 8

ROUGH NOTES BY RAVI VAKIL

So far in this series, we've avoid talking about categories fibered in groupoids in this lecture series, but I now want to introduce this perspective here, and why this is useful. So let's talk about the groupoid perspective.

One nice property: when working with categories fibered in groupoids, we can restrict naturally from global to local and get the right result (e.g. we can speicfy pairs (X_A, ϕ) : X_A flat over A, $\phi : X \to X_A$ inducing $X \xrightarrow{\sim} X_A \otimes_A k$.)

Definition. A *category cofibered in groupoids over* C is a category fibered in groupoids over C^{opp}.

Definition. A groupoid is *trivial* if there exists exactly one morphism from any object to any other. By "the" trivial groupoid we mean the one-element (one-morphism) groupoid. (Any trivial groupoid is equivalent to this.)

Remark. Artin uses (S1′). Rim uses "homogeneous groupoids". No one else uses this terminology.

Definition. A category $\mathcal{S}$ cofibered in groupoids over $\mathrm{Art}(\Lambda, k)$ is a *deformation stack* if $\mathcal{S}_k$ ist rivial, and for all $A' \to A$, $A'' \twoheadrightarrow A$, we have

(i) for all $\eta_1, \eta_2 \in \mathcal{S}_{A' \times_A A''}$, the natural map

$$\mathrm{Mor}_{A' \times_A A''}(\eta_1, \eta_2) \to_{\mathrm{Mor}_{A'}(\eta_1|_{A'}, \eta_2|_{A'})} \times_{\mathrm{Mor}_A(\eta_1|_A, \eta_2|_A)} \mathrm{Mor}_{A''}(\eta_1|_{A''}, \eta_2|_{A''})$$

is a bijection.

(ii) Given $\eta' \in \mathcal{S}_{A'}$ and $\eta'' \in \mathcal{S}_{A''}$ and $\phi : \eta'|_A \to \eta''|_A$ there exists $\phi \in \mathcal{S}'_A \times_A A''$ inducing η', η'', ϕ on restriction.

This definition is reminiscent of that of stacks.

Given $\mathcal{S}$, we write $F_S : \mathrm{Art}(\Lambda, k) \to$ **Set** for the functor of isomorphism classes of objects. Rim pointed out the following.

Proposition. Let $\mathcal{S}$ be a deformation stack. Then the associated functor is a deformation functor, hence satisfies (H1) and (H2) by the definition of deformation functor.

Proof. First we see that $F_S(k)$ is the one-element set. This is immediate, because we assumed the fiber of S over k is trivial.

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(H1) follows immediately from the second condition.

We think of (H1) as some sort of surjectivity statement, and (H2) is an injectivity statement, given (H1).

In fact, we get injectivity of (*) (that map he's been using both weeks) as long as $A = k$. In this case, the fiber product in (i) is in fact a product (as we are taking the fiber product over a one-element set). Then the result is pretty clear. $\square$

Remarks. Although being a deformation stack is formally a stronger condition than just satisfying (H1) and (H2), in real life, when you come across something satisfying (H1) and (H2), it comes from a deformation stack. Moreover, any proof (H1) and (H2) comes through really showing that it is a deformation stack.

As an example of this, consider deformations of a scheme X, Def_X. The earlier proposition actually proves the deformation stack condition.

Lemma. If $\mathcal{S}$ is the lcoal deformation problem at a point of an Artin stack, then $\mathcal{S}$ is a deformation stack.

Remark. The argument for the lemma involves the asymmetry of on $A'' \longrightarrow A$ being surjective, because we have to use the formal criterion for smoothness applied to the smmooth cover by a scheme.

We won't prove this lemma. This is Lemma 1.4.4 of a paper by MMartin Olsson called "Crystalline cohomology of stacks and Hyodo-Kato cohomology". Martin refuses this to be attributed to him, but Brian has not even seen its statement anywhere else.

Here are some more good properties of deformation functors.

Given a map $A' \to A, \eta \in \mathcal{S}_A$,

$$\{(\eta', \phi) | \eta' \in \mathcal{S}_{A'},\ \eta'|_A \xrightarrow[\sim]{\phi} \eta\ \}/\cong$$

is a pseudo-torsor over $T_S \otimes I$. Here by T_S we mean T_{F_S}.

$A' \to A, \eta' \in \mathcal{S}_{A'}, \phi \in \mathrm{Aut}(\eta'|_A), \{\phi' \in \mathrm{Aut}(\eta') : \phi'|_A = \phi\}$ is a torsor over $\mathrm{Aut}(\zeta_\epsilon) \otimes I$, ζ_ϵ is trivial deformation over $k[\epsilon]$.

You can also interpret (H4) in this category as a very natural condition, and you can understand it very geometrically.

Proposition. If $\mathcal{S}$ is a deformation stack, then $F_{\mathcal{S}}$ satisfies (H4) if and only if for $A' \twoheadrightarrow A$, and all $\eta' \in \mathcal{S}_{A'}$, the map

$$\mathrm{Aut}(\eta') \to \mathrm{Aut}(\eta'|_A)$$

is surjective.

This is actually quite easy to prove (just a few lines).

(In fancier language, in a global setting, (H4) is equivalent to saying that the $\underline{\mathrm{Isom}}$ functor is smooth at the identity.)

Why the phrase "deformation *stack*"?

This is related to the question **Why all these ring fiber products?**

Lemma. Diagrams

$$\begin{array}{ccc} A' \times_A A'' & \longrightarrow & A'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & A \end{array}$$

are in natural bijection with diagrams

$$\begin{array}{ccc} B' & \xrightarrow{q'} & B \\ {\scriptstyle q''}\downarrow & & \downarrow \\ B'' & \longrightarrow & B' \otimes_B B'' \end{array}$$

and $B \to B' \times B''$ is an injection and $q'(\ker q'')$ is an ideal of B.

This lemma is pretty straightforward to prove.

You can check that $B \to B' \times B''$ is injective iff $\operatorname{Spec} B' \coprod \operatorname{Spec} B'' \to \operatorname{Spec} B$ is scheme-theoretically surjective. (You can imagine this pictorially.)

Also, you can check that $\otimes$ corresonds to fiber products of schemes, i.e. intersections from the point of view of descent theory.

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