DEFORMATION THEORY WORKSHOP: OSSERMAN 7

ROUGH NOTES BY RAVI VAKIL

Two remaining questions remain to consider after Schlessinger's criteria: effecitivity and algebraization.

Question (effectivity). Suppose F is a deformation functor coming from a global problem, $R \in \widehat{\mathrm{Art}}(\Lambda, k)$, and $\eta \in \hat{F}(R)$, when does η come from a family over $\operatorname{Spec} R$ for the original problem?

Question (algebraization). In some situation, above answer is yes, so we have something over $\operatorname{Spec} R$, when is this induced from an "algebraic object", e.g. from something over R', of finite type over the base?

Effectivity. There no general positive answer. The main tool for positive results is the Grothendieck Existence Theorem. Let me remind you of what you've seen in the background lectures.

Theorem. Suppose $f : X \to \operatorname{Spec} A$ proper, A complete local Noetherian ring. Let $A_n = A/\mathfrak{m}^{n+1}$, $X_n = X \times_A A_n$.

You have an equivalence of categories between coherent sheaves on X and the systems of coherent sheaves on the X_n. Here's a simpler version. Given $\{\mathcal{F}_n\}$ a compatible collection of coherent sheaves on the X_n, then there exists an $\mathcal{F}$ on X coherent with $\mathcal{F}|_{X_n} = \mathcal{F}$ for all n.

Thiis gives a positive result fo effectivity in the specific case of coherent sheaves on a proper scheme.

What about deformations of an abstract scheme?

It is okay, but things go wrong for surfaces.

Specifically, it fails for K3 surfaces ($K_X = 0$ and $H^1(X, \mathcal{O}_X) = 0$). In this case, the formal deformation space is smooth of dimension 20, and there is a family over any formal neighborhood, but not over the spectrum of that ring. This caused some discussion.

The patch is to work with polarized varieties, i.e. with a choice of an ample line bundle. This follows from the full statement of Grothendieck's theorem.

Here's the idea. Take a huge multiple of the ample line bundle (so it gives an embedding into projective space, and all higher cohomology vanishes). Then the question can

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be (roughly) interpreted in terms of deforming a closed subscheme of $\mathbb{P}^n$, so you're deforming $\mathcal{I} \hookrightarrow \mathcal{O}_{\mathbb{P}^n}$, and then you're again dealing with deforming coherent sheaves and maps between them.

Algebraization. Artin restricts his attention to universal families. In that case you can get some very general positive results. He proves a positive result quite generally. This requires the base S to be of finite type over a field or an excellent Dedekind domain. But don't worry about what "excellence" means. Everything you care about is, like, totally excellent.

(Aside: By the way, "versal" means basically the same thing as "hull" except the map of functors is formally smooth.)

Definition. Let F : **Schemes**/S $\rightarrow$ **Sets** be a contravariant functor. F is *locally of finite presentation* over S (a notion which came up in passing in Max's talk earlier today) if it "commutes with limits" in a certain sense: for all filtered projective systems of affine schemes $Z_\lambda \in$ **Schemes**/S, we have $\lim_{\rightarrow} F(Z) = \lim F(\lim_{\leftarrow Z_\lambda})$. This is handy if you can prove something for Noetherian rings, and you want to prove it for all rings.

Why this definition? From EGA: if $F = h_X$ some $X \in$ **Schemes**/S, then this equivalent to $X \rightarrow S$ being locally of finite presentation.

Notation. F is a deformation functor. (R, ζ), $\zeta \in \hat{F}(R)$, $R \in \widehat{\mathrm{Art}}(\Lambda, k)$. We say ζ is *smooth over* F if the induced map $h_R \rightarrow F$ is smooth. Brian believes this is equivalent to what Artin calls versal.

Theorem. Suppose F : **Schemes**/S $\rightarrow$ **Set** is locally of finite presentation, and $\eta_0 \in F(k)$ given some $\operatorname{Spec} k \rightarrow S$ of finite type, wiith image $s \in S$. Let R be a complete lcoal Noetherian $\mathcal{O}_{S,s}$-algebra with residue field k, and suppose we have $\zeta \in F(R)$ which induces η_0 over k, and with (R, ζ) smooth over the local deformation functor corresopnding to η_0. Then there exists X of finite type over S, $x \in X$ closed, and $\eta \in F(X)$, with an isomorphsm $\hat{\mathcal{O}}_{X,x} \xrightarrow{\sim} R$ such that η maps to $\eta_n \in F(R/\mathfrak{m}_R^{n+1})$ for all n.

In general, this doesn't imply that η maps to ξ, unless of course ξ is uniquely determined by its truncations ξ_n.

Note that we snuck our effectivity into here, because we are already assuming we have our family over $\operatorname{Spec} R$.

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