DEFORMATION THEORY WORKSHOP: OSSERMAN 6

ROUGH NOTES BY RAVI VAKIL

Mori used a lower bound on the dimension of a space of morphisms (in terms of tangent and obstruction spaces) as a key technical tool to prove amazing theorems about existence of rational curves on varieties.

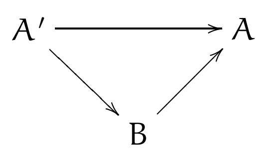
Background on obstruction theories.

Definition. $\pi : A' \to A$ in $\mathrm{Art}(\Lambda, k)$ is a *thickening* if it is surjective, with $\ker \pi \mathfrak{m}_{A'} = 0$, i.e. $\ker \pi$ has a k-vector space structure.

Definition. Given a predeformation functor F, an *obstruction theory* for F is a vector space V/k, and for all $\pi : A' \to A$ thickenings, and all $\eta \in F(A)$, an element $\mathrm{ob}(\eta, A') \in V \otimes_k \ker \pi$, such that

(i) $\mathrm{ob}(\eta, A') = 0$ if and only if there exists $\eta' \in F(A')$ such that $\eta'|_A = \eta$.

(ii) If



with $\ker(A' \to A) = I$, $\ker A' \to B = J$, then $\mathrm{ob}(\eta, B)$ is induced by $\mathrm{ob}(\eta, A')$ (using $V \otimes I \to V \otimes I/J$).

Theorem. Suppose F has a hull (R, ζ) and an obstruction theory taking values in V. Then

$$\dim \Lambda + \dim T_F \leq \dim R \leq \dim \Lambda + \dim T_F.$$

If Λ is regular, and the first inequality is an equality, then R is a complete intersection in $\Lambda[[t_1, \ldots, t_r]]$.

Lemma. Suppose $F_1 \to F_2$ is a smooth morphism of predeformation functors and we have an obstruction theory or F_2 taking values in V. Then we obtain an obstruction theory for F_1, takinng values in V.

Proof. Given $A' \to A, \eta \in F_1(A)$, set $\mathrm{ob}(\eta, A') = \mathrm{ob}(f(\eta), A')$. By smoothness, this satisfies (i), and (ii) is a diagram chase.

Date: Tuesday July 31, 2007.

Proof of theorem. The lemma reduces to the case $F = \overline{h}_R$, since by definition of a hull $\overline{h}_R \to F$ is smooth and induces an isomorphism $T_R \cong T_F$. Let $d = \dim T_R$. Schlessinger constructs R as S/J, where $S = \Lambda[[t_1, \dots, t_d]]$, so it is enough to prove that J can be generated by at most $\dim V$ elements.

By the Artin-Rees lemma, we have $J \cap \mathfrak{m}_S^n \subset J\mathfrak{m}_S$ for some n. Setting

$$A' = \Lambda[[t_1, \dots, t_j]]/(\mathfrak{m}_S J + \mathfrak{m}_S^n)$$

and

$$A = \Lambda[[t_1, \dots, t_j]]/(J + \mathfrak{m}_S^n),$$

this gives a thickening

$$0 \to I \to A' \to A \to 0$$

where $I = (J + \mathfrak{m}_S^n)/(\mathfrak{m}_S J + \mathfrak{m}_S^n) = J/\mathfrak{m}_S J$. From the quotient map $R = S/J \to A$, we have an object $\zeta_A \in \overline{h}_R(A)$, and an obstruction $\mathrm{ob}(\zeta_A, A') \in V \otimes I$ to lifting to a map $R \to A'$. We can write $\mathrm{ob}(\zeta_{A'} A') = \sum_{j=1}^{\dim V} v_j \otimes \overline{x}_j$, where the v_j form a basis for V, and $\overline{x}_j$ are images of some $x_j \in J$. We want to show that the x_j generate J. It is enough to see that the $\overline{x}_j$ generate $I = J/\mathfrak{m}_S J$, by Nakayama's Lemma.

Connsider $B = A'/(\overline{x}_i)$, this surjects onto A with kernel I', we get $\mathrm{ob}(\zeta_A, B) \in V \otimes I'$. But by functoriality, this obstruction class is 0, so we have a lift $R \to B$.

$$(1) \qquad \begin{array}{rccccc} S = & \Lambda[[t_1, \dots, t_j]] & \longrightarrow & R & & \\ & \downarrow & & \downarrow & \searrow & \\ S = & \Lambda[[t_1, \dots, t_j]] & \longrightarrow & B & \longrightarrow & A \end{array}$$

We want: $J \subset \mathfrak{m}_S J + (x_i) + \mathfrak{m}_j^n$ (recall $= \ker S \to B$) We can choose some $\phi : S \to S$ making the above diagram commute by choosing $\phi(t_i)$ appropriately: ϕ commutes with the two maps to A, so is the identity modulo $J = \mathfrak{m}_S^n$. In particular, ϕ the identity on $\mathfrak{m}_S/\mathfrak{m}_S^2$, so ϕ is an isomorphism.

So $\phi^{-1}(J) \subset J + \mathfrak{m}_S^n$, so $J \subset \phi(J) + \phi(\mathfrak{m}_S^n)$ and $\phi(\mathfrak{m}_S^n) = \mathfrak{m}_S^n$.

By commutativity of the square (1), $\phi(J) \subset \mathfrak{m}_S J + (x_i)^2 + \mathfrak{m}_s^2$.

So $J \subset \phi(J) + \mathfrak{m}_S^n \subset \mathfrak{m}_S J + (x_i) + \mathfrak{m}_s^n$. □

Example. Say X, Y are smooth varieties, $f : X \to Y$. Consider the space of deformation of the map f, i.e. Def_f. By the universal property of differentials, the tangent space is given by $H^0(X, f^* T_Y)$. It is a fact (done by Martin) that there is an obstruction theory in $H^1(x, f^* T_Y)$. If X is a curve, then $H^0 - H^1$ of $f^* T_Y$ is $\chi(f^* T_Y)$, which may be computed by Riemann-Roch, so you can really get your hands on it.

Example: deformations of a smooth surface X**.** The tangent space is $H^1(X, T_X)$, and there is an obstruction theory in $H^2(X, T_X)$. If we understand $H^0(X, T_X)$, then we can compute $H^1 - H^2$ of T_X by computing the Euler characteristic, using Riemann-Roch for surfaces. So for example, if X has finite automorphism group in characterstic 0, then $H^0(X, T_X) = 0$.

E-mail address: vakil@math.stanford.edu