DEFORMATION THEORY WORKSHOP: OSSERMAN 5

ROUGH NOTES BY RAVI VAKIL

Recall from the statement of Schlessinger's criteria:

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'') \tag{1}$$

Let's return to our proof of Schlessinger's criterion.

We've already showed that (H1)-(H3) implies that we have a hull.

All that we have to show is that if we have a hull, then we have (H1)-(H3), and then that hull+(H4) implies prorepresentability and vice versa.

Suppose F has a hull (R, ξ). Now (H3) (finite-dimensionality of the tangent space) is: if we have a hull, then $T_R \equiv T_R$, and Noetherian impiles $\dim R < \infty$.

(H1) says that if one of the maps $A', A'' \to A$ is surjective, then (1) is surjective, and (H2) says that if $A = k$, $A' = k[\epsilon]$, then we have bijectivity.

So now let's suppose that we have $p' : A' \to A$, and $p'' : A'' \to A$ in our category of Artin rings, and we assume p' is a surjection. Then we want to assert that (1) is surjective.

So all that means is if we have an object on A' and an object on A'' that restrict to the same object on A, then they should both be the restriction of some object on $A' \times_A A''$. Suppose we have $\eta' \in F(A')$, $\eta'' \in F(A'')$, both restricting to $\eta \in F(A)$. Since $\overline{h}_R \to F$ is smooth, then (by exercise) it is surjective, so there exists $u' : R \to A'$ such that $u'(\xi) = \eta'$.

Also, using the formal criterion for smoothness applied to p'', there exists $u'' : R \to A''$ with $u''(\xi) = \eta''$. Set $\zeta = u' \times_u u''(xi) \in F(A' \times_A A'')$, this lfts (η', η'') and $p'' \circ u'' = p' \circ u'$.

For (H2), assume $A = k$, $A'' = k[\epsilon]$, we want (1) injective. Suppose $\nu \in F(A' \times_A A'')$ also restricts to η' and η'', and we want $\nu = \zeta$. Keeping the same $u' : R \to A'$, we apply smoothness to $A' \times_k k[\epsilon] \to A'$ to obtain $q'' : R \to A'$ such that $u' \times q''(\zeta) = \nu$. Because $T_R \equiv T_F$, and had $u' \times u''(\xi) = \zeta$, $u'', q'' \in T_r$, so since $u''(\xi) = \zeta|_{A''} = \nu|_{A''} = q''(\xi)$, so $u'' = q''$, so $\zeta = \nu$. This

Now supose (H1)–(H4) hold. We know we have a hull (R, ξ), so we want that it prorepresents F, i.e. for all Artin rings A, we have a bijection $h_R(A) \xrightarrow{\sim} F(A)$. We'll prove this by induction on the length of A. Let $p : A' \to A$ be a small thickening, with kernel I, and suppose $h_R(A) \xrightarrow{\sim} F(A)$, and we want to conclude $h_R(A') \to F(A')$ for all $\eta \in F(A)$, have that $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$ are both pseudotorsors under $T_F \otimes I = T_R \otimes I$ (we

Date: Monday July 30, 2007.

use (H4) here!!), compatibly by functoriality But have surjection, so they msut be in bijection. Since this holds for all $\eta \in F(A)$, we have a bijection $h_R(A') \xrightarrow{\sim} F(A')$. So (R, ζ) prorepresents F by induction.

Conversely, suppose that F is prorepresentable, then (1) is always bijective, because $A' \times_A A''$ is a categorical fiber product in $\widehat{\mathrm{Art}}(\Lambda, k)$.

Let's do more examples.

More examples.

Deformations of a quotient sheaf: Let X_A be a scheme over Λ, with a quasicoherent sheaf $\mathcal{E}_\Lambda$. Write X, $\mathcal{E}$ for their restriction to k. Fix a quotient $\mathcal{E} \twoheadrightarrow \mathcal{F}$ a quasicoherent quotient.

$\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ sends A to

$$\{ \mathcal{E}_\Lambda|_\Lambda \twoheadrightarrow \mathcal{F}_A \text{ flat over } A, \text{restricting to } \mathcal{E} \twoheadrightarrow \mathcal{F} \text{ after } \otimes k\}$$

Note: no autojmorphisms to worry about; we could even have version of equality of quotients coming from equality of kernels.

Theorem. $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ is a deformation functor, and satisfies (H4).

If X_Λ is proper and $\mathcal{E}$ is coherent, then $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ satisfies (H3), so is prorepresentable.

Note for experts. For representability of global version (the "Quot scheme"), we need a projective hypothesis. But we see that the local behavior is still "scheme-like" under a properness hypothesis. This hints at algebraic spaces.

I'll sketch the proof of this theorem.

Given $A' \to A$, $A'' \to A$, and $\mathcal{F}_{A'}$, $\mathcal{F}_{A''}$ both restricting to $\mathcal{F}_A$ on A. Set $B = A' \times_A A''$, and set $\mathcal{F}_B = \mathcal{F}_{A'} \times_{\mathcal{F}_A} \mathcal{F}_{A''}$, we get a map $\mathcal{E}_B = \mathcal{E}_\Lambda|_B \to \mathcal{F}_B$ that you can check is a surjection. In fact, we have $\mathcal{E}_B \longrightarrow \mathcal{E}_{A'} \times_{\mathcal{E}_A} \mathcal{E}_{A''} \twoheadrightarrow \mathcal{F}_B$. This isn't necessarily an isomorphism, but that's okay.

This gives (H1), but we actually constructed an inverse (1), so we get (H2) and (H4) too.

This leaves (H3). You have an exercise to show that the tangent space of $\mathrm{Def}_{\mathcal{F},\mathcal{E}}$ is canonically

$$H^0(X, \underline{\mathrm{Hom}}(\mathcal{G}, \mathcal{F}))$$

, where $\mathcal{G} = \ker(\mathcal{E} \to \mathcal{F})$.

Under our extra hypothesis, this is finite-dimensional, so (H3) is satisfied.

Corollary. Given X_Λ/Λ, and $Z \subset X$, then $\mathrm{Def}_{Z,X}$ is a deformation functor, and satisfies (H4). If further X is proper over k, then (H3) is satisfied, so is prorepresentable.

Proof. Set $\mathcal{E}_\Lambda$ from the theorem to be $\mathcal{O}_{X_\Lambda}$. □

Example. Given X_Λ, Y_Λ over Λ, and $f : X \to Y$ over k, and Def_f sends A to

$$\{f_A : X_\Lambda|_A \to Y_\Lambda|_A \text{ over } A\}$$

restricting to f on k.

Corollary. If X_A and Y_A are locally of finite type over Λ, and X_A is flat over Λ, Y_Λ separated over Λ, then Def_f is a deformation functor and satisfies (H1), (H2), and (H4).

If X_Λ and Y_Λ are proper, then we also get (H3). (In fact, we don't need Y_Λ to be proper, but we have this hypothesis so it will follow from earlier results in this lecture.)

E-mail address: vakil@math.stanford.edu