DEFORMATION THEORY WORKSHOP: OSSERMAN 4

ROUGH NOTES BY RAVI VAKIL

We will now prove **Schlessinger's Theorem:** Let F be a predeformation functor. Then F has a hull iff it satisfies (H1)-(H3), and F is prorepresentable if (H1)-(H4) is satisfied.

Proposition. *Let F be a predeformation functor, and $A' \to A$ a small thickening, with kernel I. For every $\eta \in F(A)$, when the set of $\eta' \in F(A')$ restricting to η is nonempty, it has a transitive action of $T_f \otimes_k I$. This action commutes with any morphism $F' \to F$ of deformation functors.*

Also,(H4) is satisfied if and only if for all small thickenings, and all choices of $\eta \in F$, this action is also a free action as well, so it does make it into a torsor (whenever the set is non-empty).

This has been put into an exercise, along with an outline of the approach. The details won't concern us in the future, but the philosophy is worth understanding: the fact that you have this "homogeneous structure".

Definition. A surjection $p : A' \twoheadrightarrow A$ in $\operatorname{Art}(\Lambda, k)$ is *essential* if for all $q : A'' \to A'$ such that $p \circ q$ is surjective, then q is surjective.

This is a slightly abstract definition, but if we restrict outself to small thickenings, then it can be interpreted relatively concretely.

Lemma. If p is a small thickening, then p is *not* essential if and only if p has a section.

The proof is left as an exercise, broken up into a couple of parts.

Example. $k[\epsilon] \to k$, $\mathbb{Z}/p^2 \to \mathbb{F}_p$ is essential.

Given these two statements above, we can dive right into the proof.

Proposition. If (H1)-(H3) are satisfied, then F has a hull.

Proof. The proof comes in two parts. First, we construct the hull, and the second is to show that it is a hull.

Let's now construct the hull: (R, ξ), where $R \in \widehat{\operatorname{Art}}(\Lambda, k)$, $\xi \in \hat{F}(R)$, such that $\overline{h}_R \xrightarrow{\xi} F$ is formally smooth, and induces an isomorphism on tangent spaces $T_R \xrightarrow{\sim} T_F$.

Let $\mathfrak{n}$ be the maximal ideal of Λ, $r = \dim T_F$ (so $< \infty$ by (H3)). Set $S = \Lambda[[t_1, \dots, t_r]]$, and let $\mathfrak{m}$ be the maximal ideal of S.

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We'll construct R as S/J, where $J = \cap_{i\geq 2} J_i$, where what happens up to order i will be controlled by J_i. (So the J_i's are nested, $\cdots \subset \cdots \subset J_3 \subset J_2$.)

Let $\mathfrak{m}^2 + \mathfrak{n}S$. Then $S/J_2 = k[T_S^*] \cong k[T_F^*]$. We can write tihs (non-canonically) as $k[\epsilon] \times \cdots \times k[\epsilon]$ (r times), by choosing a basis of T_F^*.

Define $R_2 = S/J_2$.

We use (H2) to construct a $\xi_2 \in F(R_2)$ inducing a bijection $T_{R_2} \xrightarrow{\sim} T_F$.

Now we want to induct.

Suppose we have $R_{i-1} = S/J_{i-1}$, and $\xi_{i-1} \in F(R_{i-1})$.

We'll choose J_i to be minimal among J satisfying:

- $\mathfrak{m}J_{i-1} \subset J \subset J_{i-1}$
- ξ_{i-1} can be lifted to an element of $F(R_1)$

In order for "minimal" to make sense, we need to make sure that these two conditions are preserved under arbitrary intersections. The first certainly is. So let's check the second assuming the first; this is actually a little bit involved.

Note: J satisfying the first condition corresponds to vector subspaces of $J_{i-1}/\mathfrak{m}J_{i-1}$, which is finite-dimensional. Hence if you want to check if some collection of subspaces of a finite-dimensionnal space, and you want to check if the intersection of an arbitrary (potentially infinite) number of subspaces in your collection is also in the collection, then you need only check that the intersection of two of elements of your collection is also in the collection.

Thus it is enough to check pairwise intersections.

Suppose J, K satisfy our conditions. We'll show that $J \cap K$ does too. Again using the finite-dimensionality of $J_{i-1}/\mathfrak{m}J_{i-1}$, we can replace K without changing $J \cap K$ so that $J + K = J_{i-1}$. So now we've set ourselves up to use Schlessinger's criteria.

Then $S/J \times_{S/J_{i-1}} S/K \cong S/(J \cap K)$, so by (H1), we have some element of $F(S/J \cap K)$ restricting to ξ_{i-1}, which means $J \cap K$ satisfies our conditions.

Set $J = \cap_i J_i$, and $R = S/J$. We have the advantage that we can think of S/J_i for each i; we'll give the name R_i to this quotient of R.

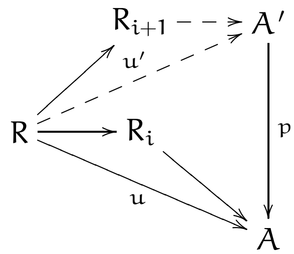
You can check that $\mathfrak{m}^i \subset J_i$, so we have $R = \varprojlim R/J_i$, and $\xi = \varprojlim \xi_i$ makes sense.

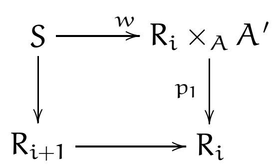
Now let's prove that (R, ξ) *is* a hull. This will use the results stated at the beginning of the lecture.

The isomorphism $T_R \xrightarrow{\sim} T_F$ is immediate from the choice of ξ_2, smoothness is harder.

Fix $pA' \to A$ a small thickening, $\eta' \in F(A')$ such that $p(\eta') = \eta \in F(A)$, and $u : R \to A$ such that $u(\xi) = \eta$. We want a lift $u' : R \to A'$ such that $u'(\xi) = \eta'$. First construct any u' lifting u.

Since A is an Artin ring, u factors through $R \to R_i$ for some i. We want to fill in the lower dashed arrow, from which it suffices to fill in the upper dashed arrow.





p_1 is a small thickening. If we have a section, no problem. If not, p_1 is *essential* (i.e. no section!), so we choose w as above, which must be surjective. It is enough to show that $\ker w \sup J_{i+1}$. This follows from (H1).

So we have some u', and we want to have $u'(\xi) = \eta'$. But we have compatible transitive actions of $T_F \otimes I \cong T_R \otimes I$ of $T_F \otimes I \cong T_R \otimes I$ of $F(p)^{-1}(\eta)$ and $h_R(p)^{-1}(\eta)$, which consists of those $R \to A'$ such that $A \to A$ sends ξ to η.

By this transitivity, there exists some $\tau \in T_F \otimes I$ sending $u'(\xi)$ to η'. Then we can modify u' by τ, and we'll have the desired u' lifting u, sending ξ to η'.

So we've done the first half of Schlessinger! We've actually done the bulk of what we need to do for the entire proof.

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