DEFORMATION THEORY WORKSHOP: OSSERMAN 3

ROUGH NOTES BY RAVI VAKIL

Recall the definition of a predeformation functor: a predeformation functor is a (covariant) functor $F : \mathbf{Art}(\Lambda, k) \to \mathbf{Sets}$ such that $F(k)$ is the one-point set.

Last day, we saw the definitions of Schlessinger's criteria for a predeformation functor F.

Given $A' \to A$, $A'' \to A$, we have:

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A''). \tag{1}$$

- (H1) (1) is surjective when $A'' \twoheadrightarrow A$ (or equivalently, for small thickenings)
- (H2) (1) is bijective when $A'' = k[\epsilon]$, $A = k$ (or equivalently, for small thickenings)
- (H3) T_F is finite-dimensional.
- (H4) (1) is bijective whenever $A' = A''$ and they both surject onto A.

Remarks.

- Fiber products of rings may seem strange. We'll come back to this.
- (H1) and (H2) are essentially always satisfied
- (H3) tends to be related to properness.
- (H4) is related to automorpihsms. We'll see something about this soon, but for now we'll leave this vague.

Definition. A deformation functor is a predeformation functor that also satisfies (H1) and (H2).

A repeated note from yesterday in new language: (H3) makes sense for any deformation functor.

Definition. Given $(X_A, \phi) \in \mathrm{Def}_X(A)$ an automorphism of (X_A, ϕ) (or less precisely, an infinitesimal automorphism of X_A) is an automorphism of X_A over A, commuting with ϕ.

This was implicit in last day's discussion, when we defined isomorphisms between two elements of $\mathrm{Def}_X(A)$.

Theorem. Let X be a scheme over k, and Def_X the functor of deformations of X. Then

(i) Def_X is a deformation functor, i.e. satisfies (H1) and (H2).

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(ii) Def_X satisfies (H3) if X is proper. (*Not* only if, as we've seen in the case when X is smooth and affine!)
(iii) Def_X satisfies (H4) if and only if for all $A' \to A$ and small thickenings, and $(X_{A'}, \phi)$ over A', every automorphism of $(X_{A'}|_A, \phi|_A)$ is the restriction of an automorphism of $(X_{A'}, \phi)$.

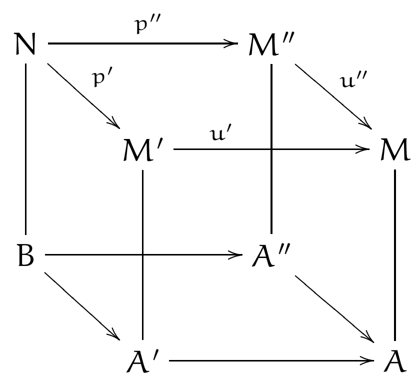
In particular, if $H^0(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)) = 0$, then we will see that none of these $(X_{A'}, \phi)$ have any non-trivial automorphisms.

Corollary. If X is proper, then $\mathrm{Def}\, X$ has a hull. If further $H^0(X, \mathcal{H}om(\Omega^1_{X/k}, \mathcal{O}_X)) = 0$ then $\mathrm{Def}\, X$ is prorepresentable.

Example. If X is a smooth proper curve, then Def_X has a hull, and $\mathrm{Def}\, X$ is prorepresentable if $g \geq 2$.

Here is a basic lemma about flatness. It looks like a mess, but it's handy.

Lemma. Consider the cube



which is a compatible commutative diagram of ring and module homomorphisms and such that

- the top square and bottom square are fiber diagrams.
- $A'' \to A$ is surjective with nilpotent kernel, and
- u' induces an isomorphism $M' \otimes_{A'} A \cong M$ and similarly for u''

Then N is flat over B, and p' induces $N \otimes_B A \xrightarrow{\sim} M'$, and similarly for p''. Also, in the same situation, if we have L a B-module a B-module, and $q : L \to M'$ and $q'' : L \to M''$ such that q' induces an isomorphism $L \otimes_B A' \to M'$ (but with no condition on M'') then $q' \times q'' : L \to N$ is an isomorphism.

Why am I telling you this? Because this is useful if you're trying to prove (H1) in the affine case.

Note: this is more general than is necessary for Schlessinger, as these A, A', A'' aren't assumed to be Artin rings. The proof is much easier in the case of Artin rings (as over an Artin ring, flat = free), but we'll need this more general case next week.

The proof of the lemma is left as a serious exercise; it will use the local criterion of flatness.

Let's see what this tells us about schemes.

The general proposition is as follows.

Proposition. Given $A' \to A$, $A'' \to A$, where $A'' \to A$ is surjective with nilpotent kernel, write $B = A' \times_A A''$. Then:

(i) Given X' and X'' flat over A' and A'', and an isomorphism $\phi : X'|_A \xrightarrow{\sim} X''|_A$, there exists some Y flat over B, with maps $\phi' : X' \to Y$ and $\phi'' : X'' \to Y$ inducing isomorphisms $X' \longrightarrow Y|_{A'}$ and $X'' \longrightarrow Y|_{A''}$.

In other words, we're creating a Y that is flat over B, that "extends X' and X''". ϕ is recovered by

$$\phi = \phi''|_A \circ (\phi')^{-1}|_A.$$

(ii) Given Y_1 and Y_2 flat over B, the natural map

$$\mathrm{Isom}_B(Y_1, Y_2) \to \mathrm{Isom}_{A'}(Y|_{A'}, Y_2|_{A'}) \times_{\mathrm{Isom}_A(Y_1|_{A''}, Y_2|_{A''})} \mathrm{Isom}_{A''}(Y|_{A''}, Y_2|_{A''})$$

is a bijection.

Proof. (i) We'll construct Y on the same topological space on X'. So all we have to do is construct the sheaf of rings $\mathcal{O}_Y$ on this space. We'll do this by brute force. We identify the maps X'' and $X''|_A$, and also $X'|_A$ using ϕ, and write $i : X'|_A \to X'$. Set $\mathcal{O}_Y(U) = \mathcal{O}_{X'}(U) \times_{\mathcal{O}_{X'}|_A(i^{-1}(U))} \mathcal{O}_{X''}(i^{-1}(U))$. Yowtch! Martin gave an alternative formulation of this expression, but it seems messy no matter how you slice it.

We an apparently check that this is a sheaf.

The lemma says that this $\mathcal{O}_Y$ is flat over B, and that it recovers $\mathcal{O}_{X'}$ and $\mathcal{O}_{X''}$ on restriction to A' and A'' respectively.

We can then check that it defines a scheme structure. This can be done by using the fact that the module fiber product commutes with localization.

(ii) is similar, using the second part of the lemma. This is omitted for the sake of time and patience, but details will be given in the notes.

Proof of Theorem about deformations of schemes.

(i) *(H1) and (H2) are satisfied.*

(H1) follows from part (i) of the proposition using a diagram chase. You have to be a bit careful with the rigidifying maps. There are no new ideas.

(H2) uses part (ii) of the proposition as well, and it also uses that $A = k$, so the ϕ in the definition of Def_X rigidifies all of the isomorphisms. This takes some writing out of details.

(ii) will be discussed in Martin's lectures later.

(iii) *(H4) is satisifed if and only if the automorphisms extend.* This was described verbally. It is kind of like the proof of part (i).

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