DEFORMATION THEORY WORKSHOP: OSSERMAN 2

ROUGH NOTES BY RAVI VAKIL

Today I want to give some examples of deformation functors we'll be considering. For "nice" global moduli functors, it works well to simply restrict to $\mathrm{Art}(\Lambda, k)$ to obtain predeformation functors.

Here's an example of that to start off.

Example: Deformations of subschemes of a given scheme. We'll put no hypotheses on the schemes, and as we go, we'll add hypotheses to get good behavior.

Let X_Λ be a scheme over $\operatorname{Spec} \Lambda$, and write X for $X_\Lambda|_{\operatorname{Spec} k}$, so X is a scheme over k.

Let $Z \subset X$ be a closed subscheme. The predeformation functor

$$\mathrm{Def}_{Z,X} : \mathrm{Art}(\Lambda, k) \longrightarrow \mathbf{Set}$$

is defined by $A \mapsto Z_A \subset X_\Lambda|_{\operatorname{Spec} A}$ where Z_A is a closed subscheme, flat over A, such that $Z_A|_{\operatorname{Spec} k} = Z$. (Here this is *equality* as closed subschemes, not just an abstract isomorphism. Also, "restriction to $\operatorname{Spec} A$" means pullback — A isn't necessarily a quotient of Λ.)

Note this is precisely what you get when you take a global functor and restrict it to the category of Artin rings.

Sometimes, simple restriction of functors isn't so good.

Example: Deformations of an abstract scheme. Fix X/k. Def_X is defined by $A \mapsto$ the set of (X_A, ϕ) such that X_A is flat over A, ans such that $\phi : X \to X_A$ induces a fiber diagram

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X_A \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \hookrightarrow & \operatorname{Spec} A \end{array}$$

up to isomorphism, where you have to figure out what "isomorphism of such diagrams" means.

Note: If we naively restricted functors, we still get a predeformation functor, but its behaviour will be worse.

(In fact, the naivete comes not from restricting such functors, but because such functors are nasty to begin with. By the end of the workshop, we'll replace these by a better notion.)

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Example: Deformations of a quasicoherent sheaf. This will mix the properties of the two previous examples. Fix X_Λ over $\operatorname{Spec}\Lambda$. Set $X = X_\Lambda|_k$, fix $\mathcal{E}$ a quasicoherent sheaf on X_Λ. (Feel free to consider the case where $\mathcal{E}$ is a vector bundle.)

Define $\operatorname{Def}_{\mathcal{E}}$ by $A \mapsto$ those $(\mathcal{E}_A, \phi)$ where $\mathcal{E}_A$ is a quasicoherent sheaf on $X_\Lambda|_A$, flat over A, $\phi : \mathcal{E}_A \to \mathcal{E}$ inducing $\mathcal{E}_A \otimes_A k \xrightarrow{\sim} \mathcal{E}$, modulo equivalence. A good exercise: what's the equivalence?

1. Flavors of representability

Following Schlessinger, we'll introduce two notions in the direction of representability: prorepresentability and hulls.

Definition. Given $F : \operatorname{Art}(\Lambda, k) \to \textbf{Set}$, let $\widehat{\operatorname{Art}}(\Lambda, k)$ be the category of complete local Noetherian Λ-algebras

(Unimportant technical point: any element of this category is a limit of Artinian guys, i.e. elements of $\operatorname{Art}(\Lambda, k)$. However, the converse is not true: a limit of Artinian guys need not be Noetherian.)

$\hat{F}(R) = \varprojlim F(R/\mathfrak{m}^n)$.

We say F is pro-representable if this new functor $\hat{F}$ is representable.

Warning. If we start with a global moduli problem, $\hat{F}$ is not necessarily obtained by simply considering families over R. Given a family of R, we get a compatible family of $R/\mathfrak{m}^n$; but if we get a compatible family, it isn't clear that it comes from a family over R. This is the question of *effectifizability*, and we'll return to this issue later.

Definition. Given $F, F' : \operatorname{Art}(\Lambda, k) \to \textbf{Set}$. A morphism of functors (a natural transformation): $f : F \to F'$ is smooth (I'd prefer: formally smooth) if every surjection $A \twoheadrightarrow B$, the map $F(A) \to F(B) \times_{F'(B)} F'(A)$ is surjective.

Exercise: Guess whawt it means for a morphism of functors to be unramified or étale.

Recall that T_F, the tangent space of F, is $F(k[\epsilon])$.

Notation. Given $R \in \widehat{\operatorname{Art}}(\Lambda, k)$, describe $h_R : \widehat{\operatorname{Art}}(\Lambda, k) \to \textbf{Set}$ is the functor of points [horrible notation] of $\operatorname{Spec} R$, where $h_R(R')$ is defined to be $\operatorname{Mor}(R, R')$. Let $\overline{h}_R$ is the restriction to $\operatorname{Art}(\Lambda, k)$.

Definition. Let F be a predeformation functor. A pair (R, η) with $\eta \in \hat{F}(R)$ is a *hull* for F if $\overline{h}_R \to F$ is smooth, and $T_{\hat{h}_R} \to T_F$ is an isomorphism.

(Side point: the fact that the tangent map is linear comes from the fact that the linear structure on those two sets arose from certain natural diagrams.)

Proposition (Any two hulls are isomorphic, up to non-unique isomorphism). If (R, η) and (R', η') are hulls for F then $(R, \eta) \cong (R', \eta')$.

This is left as an exercise. The first part of the exercise will be making precise what $\cong$ means in the statement of the result.

Note: prorepresentable implies the existence of a hull.

We can now state Schlessinger's criterion.

2. Schlessinger's criterion

This gives a criterion for testing when a functor is prorepresentable or has a hull.

These are a little opaque, but they become clearer later when we discuss categories fibered in groupoids.

However, the criterion is extremely effective, in that you can often sit down and calculate whether a functor satisfies them.

Definition. A surjective map $f : A \twoheadrightarrow B$ in $\mathrm{Art}(\Lambda, k)$ is a *small thickening* if

(i) $\ker f \cong k$, or equivalently,
(ii) $\mathfrak{m}_A \ker f = 0$ and $\ker f$ is principal.

It is easy to check that any surjective map in our category can be factored into a sequence of small thickenings. This might be a good exercise.

Given $A' \to A$, $A'' \to A$, we have:

$$F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'') \tag{1}$$

Theorem (Schlessinger). If F is a predeformation functor, consider:

(H1) (1) is surjective when $A'' \twoheadrightarrow A$ (or equivalently, for small thickenings)
(H2) (1) is bijective when $A'' = k[\epsilon]$, $A = k$ (or equivalently, for small thickenings)
(H3) T_F is finite-dimensional.
(H4) (1) is bijective whenever $A' = A''$ and they both surject onto A.

Then (H1)–(H3) are equivalent to F having a hull, and (H1)–(H4) are equivalent to F being prorepresentable.

Note that (H3) makes sense when (H2) is satisfied, thanks to Martin's lecture.

Next time we'll talk about them further, and check them for a particular case.

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