DEFORMATION THEORY WORKSHOP: OSSERMAN 1

ROUGH NOTES BY RAVI VAKIL

This lecture series will be about functors of Artin rings, and in particular representability and Schlessinger's criterion.

We've seen two extremes of understanding schemes by looking at maps to that scheme. We've seen Yoneda's lemma, that we can completely recover a scheme (up to unique isomorphism!) from the maps to it (the "funtor of points"), in Max's letter. On the other hand, in Martin's lecture, we've seen that (under mild hypotheses) the tangent space is recovered by looking at maps from $\operatorname{Spec} k[\epsilon]/\epsilon^2$. (We will always use the convention that $\epsilon^2 = 0$.

We'll now talk about something in between these two extremes (very global and tangent-space-local). It will still by quite local: maps $\operatorname{Spec} A \to X$, where A is a local Artin ring.

From a moduli perspective, we're studying families over $\operatorname{Spec} A$. We'll the restriction $\operatorname{Spec} A/\mathfrak{m} = \operatorname{Spec} k$. That's just a single point, so you might think that not much might go on over a point, but you're wrong! Although it is true: topologically, nothing can go on; all the action is on the level of the sheaf of rings. These are called "infinitesimal thickenings". The information we'll get if the moduli space is a scheme (and, as we'll see, in some important more general circumstances) we'll get the complete local ring.

1. Recovering complete local rings

We define some temporary notation.

$\mathbf{Art}_k$ is the category of local Artin rings with residue field k, over $\operatorname{Spec} k$. Morphisms are supposed to be compatible with the map to $\operatorname{Spec} k$.

Given a locally Noetherian scheme X, and $x \in X$.

Let $F_{X,x} : \mathbf{Art}_k \to \mathbf{Set}$ given by

$$(X \twoheadrightarrow k) \mapsto \{ f : \operatorname{Spec} A \to X \text{ such that } f \circ (\operatorname{Spec} k \hookrightarrow \operatorname{Spec} A) = (\operatorname{Spec} k \xrightarrow{x} X) \}$$

So what kind of data is encoded in this functor?

Proposition. Given X locally Noetherian, $x \in X$, the canonical map $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to X$ induces

$$F_{\operatorname{Spec} \hat{\mathcal{O}}_{X,x}, X}(A) \to F_{X,x}(A)$$

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is a bijection for each Artin ring A (or an *isomorphism of functors*), and any complete complete local Noetherian ring R with $\operatorname{Spec} R \to X$ inducing such a bijection (i.e. with the same closed point etc.) is canonically isomorphic to the complete local ring.

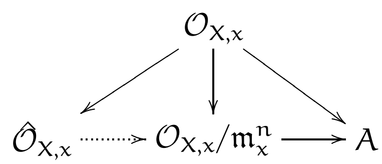
Caution. We should say what the topology is on R; it is the $\mathfrak{m}$-adic topology.

Remark. The last sentence is anticipating the notion of prorepresentability, which we'll talk about tomorrow.

Proof. The first statement, meaning the bijectivity of the map, is equivalent to saying that any map $\operatorname{Spec} A \to X$ over $\operatorname{Spec} k$ with image x factors uniquely through $\operatorname{Spec} \hat{\mathcal{O}}_{X,x} \to X$.

It is an easy exercise that it factors uniquely through $\operatorname{Spec} \mathcal{O}_{X,x}$. (Indeed that's true for any lcoal ring A.) One way to prove this is to restrict to affines, and think about rings.

So we can now think about rings. We need $\mathcal{O}_{X,x} \to A$ to factor uniquely through $\mathcal{O}_{X,x} \to \hat{\mathcal{O}}_{X,x}$. Because A is an Artin ring, the image of the maximal ring in A, the image of the maximal ideal will be nilpotent. Thus some power of it is 0. Thus some power of it maps to 0 in A. We'll write $\mathfrak{m}_x$ for the maximal ideal of $\mathcal{O}_{X,x}$. Thus some power (say the nth of it is 0 in the Artin ring. Thus $\mathcal{O}_{X,x} \to A$ factors through $\mathcal{O}_{X,x}/\mathfrak{m}_x^n$. So we get the factorization from



(That was written sloppily.)

That concludes the proof of bijectivity.

We now show that any two such are isomorphic. This is basically the same idea as in Yoneda's lemma. We'll make this precise tomorrow.

Execise. Figure this out.

suppose $R/\mathfrak{m}_R^n$ also gives such a bijection. Using a Yoneda-type argument, we construct compatible maps $\hat{\mathcal{O}}_{X,x} \to R/\mathfrak{m}_R^n$, and $R \to \hat{\mathcal{O}}_{X,x}/\mathfrak{m}_X^n$, we construct isomorphisms to and from $\hat{\mathcal{O}}_{X,x}$ and R.

Next we deal with the second part. □

Let me try to give you some sense of what this actually means in a practical way.

Remarks. What data can you recover from the formal lcoal ring $\hat{\mathcal{O}}_{X,x}$? Here are some possibilites.

(1) $\dim$ of X at x.

(2) the "singularity type" of X of x (suitably defined), something similarl to the local ring of an analytic space.

(3) For example, there is the Cohen theorem, that if X is smooth over k of dimension n, then $\hat{\mathcal{O}}_X \cong k[[x_1, \dots, x_n]]$. (We are still assuming that k is the residue field at X.) This is already mentioned in Hartshorne somewhat more analogous to the situation for complex manifolds that every point has a neighborhood isomorphic to complex n-space.

(4) e.g. $y^2 = t^3 - t^2$ gives us the same formal local ring as if you took the union of the axes. *Exercise.* Show that $k[[y,t]]/(y^2 + t^2 - t^3) \cong k[[u,v]]/(uv)$.

(5) e.g. the cusp. $y^2 = t^3$, get $k[[x,t]]/(y^2 - t^3)$ is not congruent to $k[[s]]$, even though you have "a homeomorphism". *Exercise.* Show this.

These exercises might be in Hartshorne as examples.

2. The functors of interest

We work in a relative setting: we'll fix Λ a complete lcoal Noetherian ring with residue field k, and we'll consider $\mathbf{Art}(\Lambda, k)$ of Artin loca Λ-algebras with residue field k.

Why introduce this Λ?

- If you want to work over a field, you'll take Λ_k.
- If you want to work in mixed characterstic, and k is perfect, Λ might be the Witt vectors.
- And you might be working with a moduli space that naturally is defined over a more complicated base, e.g. $\operatorname{Spec} \mathbb{Z}$.

We introduce some nonstandard terminology: A predeformation functor is a (covariant) functor $F : \mathbf{Art}(\Lambda, k) \to \mathbf{Sets}$ such that $F(k)$ is the one-point set.

Roughly, these can arise by considering families over $\operatorname{Spec} A$ restricting to a fixed object over $\operatorname{Spec} k$. But they come up in other ways.

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