DEFORMATION THEORY WORKSHOP: OLSSON 8

ROUGH NOTES BY RAVI VAKIL

The title of today's talk is: "The cotangent complex: an overview."

Suppose $f : X \to S$ a morphism of schemes.

We define

$$GF(\mathcal{O}_X) := \qquad \begin{array}{ccc} f^{-1}\mathcal{O}_S\{F(\mathcal{O}_X)\} & \overset{\pi}{\twoheadrightarrow} & \mathcal{O}_X \\ \uparrow & \nearrow & \\ f^{-1}\mathcal{O}_S & & \end{array}$$

Here G is defined as the adjoint of the forgetful functor

$$F : (\ f^{-1}\mathcal{O}_S\textbf{-alg}) \to (\textbf{sheaves of sets})$$

Then we define

$$\tau_{\geq -1}|_{X/S} := \left(I/I^2 \to \Omega^1_{GF(\mathcal{O}_X)/f^{-1}\mathcal{O}_S} \otimes \mathcal{O}_X \right)$$

where $I = \ker(\pi)$.

Then I dualized this, which I'll describe in a fancy way.

Theorem. For any quasicoherent $\mathcal{O}_X$-module I

$$ch(\tau_{\leq 0}(R\,\underline{\mathrm{Hom}}(\tau_{\geq 1}|_{X/S}, M)[1])) \cong \underline{\mathrm{Exal}}_S(X, M).$$

We call $\tau|_{\geq 1}L_{X/S}$ is called the *truncated cotangent complex*.

Here is the construction of the full cotangent complex $\boxed{L_{X/S}}$. It is not very enlightening, but I'll say it in case you are curious.

Given $n \geq 0$, apply $GF \cdots GFGF$ (where there are $n+1$ copies of G) to $\mathcal{O}_X$. Call this algebra $\mathcal{A}_n$. This is an $f^{-1}\mathcal{O}_S$-algebra, and there is a natural surjection $\mathcal{A}_n \twoheadrightarrow \mathcal{O}_X$. What kind of object is this $\mathcal{A}_\bullet$? Well, this is a *simplicial* $f^{-1}\mathcal{O}_S$*-algebra*. This means that there are a bunch $(n+2)$ of maps $\mathcal{A}_{n+1} \to \mathcal{A}_n$, and a bunch $(n+1)$ of maps in the other direction, and these have to satisfy some properties. Well, by adjointness, we have maps $a : GF \to \mathcal{A}$, and $b : id \to FG$. How we get the d_i? It is the map induced by using a, and crossing out the ith copy of GF.

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The compatibilities are really immediate from what we've done, and come only from facts about adjoints.

So we apply the Ω functor, to get a simplicial $\mathcal{O}_X$-module

$$\tilde{L}_\bullet := \Omega_{\mathcal{A}_\bullet/f^{-1}\mathcal{O}_S} \otimes \mathcal{O}_X.$$

From this, we get a complex as follows. they've got the same terms, and we take alternating sums of the differentials. The fact that it is a complex again is immediate formally (from the fact that F and G are adjoints).

Remark. This is an actual complex! It is not just an element of the derived category! It is a complex of *flat* $\mathcal{O}_X$-modules! It is huge! It has some nice properties, which we'll give now!

(i) $\mathcal{H}^i(L_{X/S})$ is quasicoherent, and coherent if S is locally Noetherian and f is of finite type. (We saw this for the first two terms by our discussion of the truncated cotangent complex.)

(ii) Suppose you have a commtuative diagram

$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ {\scriptstyle f'}\downarrow & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{\nu} & Y \end{array}$$

then there is a base-chagne morphism $u^*L_{X/Y} \to L_{X'/Y'}$. If (*) is Cartesian and "tor-independent" (e.g. if f or ν is flat) then $u^*L_{X/Y} \to L_{X'/Y'}$ is a quasiisomorphism.

I'd prefer not to define tor-independent.

Furthermore, $f'^*L_{Y'/Y} \oplus u^*L_{X/Y} \to L_{X'/Y}$ is a quasiisomorphism.

(iii) (This is the key point where you need more than the usual truncated complex that we have all come to love in the last few lectures.) If we have $X \xrightarrow{f} Y \xrightarrow{g} Z$ then there is a distinguished triangle

$$f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^*L_{Y/Z}[1]$$

in the derived category. (For readers not familiar with the derived category — i.e. almost all of you — think of this as being an exact sequence of complexes. That's right except for the third map. But we *will* have a long exact sequence whenever we apply a functor (left or right, I forget).)

(iv) $\tau_{\geq 1}L_{X/Y}$ is equal our earlier $\tau_{\geq -1}L_{X/Y}$. This is because our earlier discussion was precisely the same construction just for the first two terms.

Remarks. a) $\mathcal{H}^0(L_{X/Y}) = \Omega^1_{X/Y}$. That actually follows from our earlier <u>Exal</u> discussion.

b) If f is smooth then $L_{X/Y} \to \Omega^1_{X/Y}$ is a quasiisomorhpism.

c) If $X \hookrightarrow Y$ is a local complete intersection closed immersion, then $L_{X/Y} \to (I/I^2)[1]$ is a quasiisomorphism.

Magically, it really solves a lot of problems.

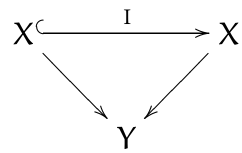
Theorem (Illusie).

$$\mathrm{ch}(\tau_{\geq -1}(R\,\underline{\mathrm{Hom}}(L_{X/Y}, I)[1]) \cong \underline{\mathrm{Exal}}_Y(X, I).$$

This implies (by "taking global sections")

$$\mathrm{Ext}^1(L_{X/Y}, I) \cong \mathrm{Exal}_Y(X, I).$$

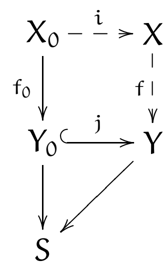
and $\mathrm{Ext}^0(L_{X/Y}, I) = \mathrm{Hom}(\Omega^1_{X/Y}, I)$ is in bijection with the automorphism group of



(This is basically the universal property of differentials, as we've seen before.)

Let's use this big machine!! We'll return to two examples from last week, except now we can turbocharge them.

Problem. Consider the diagram



Here j is a closed immersion defined by a square-zero ideal J.

Fill in the diagram as indicated with i squrae-zero such that $f_0^*J \longrightarrow \ker(\mathcal{O}_X \to \mathcal{O}_{X_0})$.

Solution. $X_0 \to Y_0 \to Y$ induces

$$f_0^* L_{Y_0/Y} \to L_{X_0/Y} \to L_{X_0/Y_0} \to f_0^* I L_{Y_0/Y}[1]$$

which gives a long exact sequence

$$0 \longrightarrow \mathrm{Ext}^0(L_{X_0/Y_0}, f_0^*J) \longrightarrow \mathrm{Ext}^0(L_{X_0/Y}, f_0^*J) \longrightarrow \mathrm{Ext}^0(L_{Y_0/Y}, f_0^*J) \longrightarrow$$

$$\mathrm{Ext}^1(L_{X_0/Y_0}, f_0^*J) \longrightarrow \mathrm{Ext}^1(L_{X_0/Y}, f_0^*J) \longrightarrow \mathrm{Ext}^1(L_{Y_0/Y}, f_0^*J) \longrightarrow$$

$$\mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J) \longrightarrow \cdots$$

We now use Illusie's theorem identifying many of these groups.

$$0 \longrightarrow \mathrm{Ext}^0(L_{X_0/Y_0}, f_0^*J) \longrightarrow \mathrm{Ext}^0(L_{X_0/Y}, f_0^*J) \longrightarrow 0 \longrightarrow$$

$$\mathrm{Ext}^1(L_{X_0/Y_0}, f_0^*J) \longrightarrow \mathrm{Exal}_Y(x_0, f_0^*J) \longrightarrow \mathrm{Hom}(f_0^*J, f_0^*J) \xrightarrow{\partial}$$

$$\mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J) \longrightarrow \cdots$$

Then we have to think a little.

Theorem. (i) There exists an obstruction $o(f_0) = \partial(id) \in \mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J)$ whose vanishing is necessary and sufficient for a solution to the problem.

(ii) If $o(f_0) = 0$, then the set of isomorphism classes of solutions form a torsor under $\mathrm{Ext}^1(L_{X_0/Y_0}, f_0^*J)$

(iii) $\mathrm{Aut} = \mathrm{Ext}^0(L_{X_0/Y_0}, f^*J)$.

The advantage of using this derived functor machinery rather than cocycle machinery is that you can now deal with things that are locally obstructed. Our earlier discussion dealing with deforming smooth varieties, for example, dealt with things with no local obstructions. (Recall that affine smooth varieties had no infinitesimal deformations.)

Problem. Consider the diagram.

$$\begin{array}{ccc} X_0 & \overset{i}{\hookrightarrow} & X \\ {\scriptstyle f_0}\downarrow & & \downarrow{\scriptstyle f} \\ Y_0 & \overset{j}{\hookrightarrow} & Y \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle g} \\ Z_0 & \overset{k}{\hookrightarrow} & Z \end{array}$$

Find the dotted arrow.

You can think of Y as a moduli space, and Z as our base.

Theorem (Illusie).

There is a class $o(f_0) \in \mathrm{Ext}^1(f_0^*L_{Y_0/Z_0}, I)$ such that f exists if and only if $o(f_0) = 0$. If $o(f_0) = 0$, then the set of maps f is a torsor under $\mathrm{Ext}^0(f_0^*L_{Y_0/Z_0}, I)$.

Martin then gave a very quick sketch of the proof.

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