DEFORMATION THEORY WORKSHOP: OLSSON 7

ROUGH NOTES BY RAVI VAKIL

Last day, we stated the following theorem.

Theorem. $\mathrm{ch} : \tilde{C}^{[-1,0]}(T) \to$ **(Picard stacks)** is an equivalence of 2-categories.

What is an equivalence of 2-categories? Answer: it is equivalent to the content of the next three lemmas.

Lemma. Let $\mathcal{P}$ be a Picard stack. Then there exists $K \in C^{[-1,0]}(T)$ and an equivalence $\mathrm{ch}(K) \xrightarrow{\sim} \mathcal{P}$.

Lemma. $K, L \in C^{[-1,0]}(T)$ and let $F : \mathrm{ch}(K) \to \mathrm{ch}(L)$ be a morphism of Picard stacks. Then there exists a quasiisomorphism $k : K' \to K$ and a morphism $l : K' \to L$ such that $F \cong \mathrm{ch}(l) \circ \mathrm{ch}(k)^{-1}$.

$$\begin{array}{ccc} \mathrm{ch}(K') & \xrightarrow{k} & \mathrm{ch}(K) \\ \downarrow l & \swarrow F & \\ \mathrm{ch}(L) & & \end{array}$$

In particular, if $K \in \tilde{C}^{[-1,0]}(T)$ then any morhpism $f : \mathrm{ch}(K) \to \mathrm{ch}(L)$ is isomorphism to $\mathrm{ch}(f)$ for some $f : K \to L$. (This isn't trivial; it is an exercise dealing with injectives.)

Sketch of proof of lemma. This is mainly one enormous construction.

Choose data $\{(U, k, l_i, \sigma_i)\}_{i \in I}$ such that

a) $U_0 \subset T$ open set

b) $k_i \in K^0(U_i)$, $l_i \in L^0(U_i)$, $\sigma = F(k_i)$.

c) the map $K'^0 := \oplus_{i \in I} \mathbb{Z}_{U_i} \to K^0$ is surjective.

$$K'^{-1} := K^{-1} \times_{K^0} K'^0$$

.

Define $l : K' \to L$ by:

$l^0 : K'^0 \to L^0$, $\mathbb{Z}_{U_i} \to L^0$ given by l_i.

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$l^{-1} : K'^{-1} \to L^{-1}$ $(v, (U_i, k_i, l_i, \sigma_i)) \in K^{-1}$.

Maps to the element $t \in L^{-1}$ such that

$$\begin{array}{ccc} F(0) & \xrightarrow{F(v)} & F(k_i) \\ \downarrow{\scriptstyle \sim} & & \downarrow{\scriptstyle \sigma_i} \\ 0 & \xrightarrow{t} & l_i \end{array}$$

We now have to check that we get an isomorphism with $\sigma : F \xrightarrow{\sim} ch(l) \circ ch(k)^{-1}$. There are a million details to check. □

Now there is a third lemma.

Lemma. K_1 and K_2 are in $\tilde{C}^{[-1,0]}(T)$. For two morphisms of complexes $f_1, f_2 : K_1 \to K_2$ with associated morphisms $F_1, F_2 : ch(K_1) \to ch(K_2)$ and any isomorphism $H : F_1 \to F_2$, there exists a unique homotopy $h : K_1^0 \to K_2^{-1}$ such that $H = ch(h)$.

Let me just give you the idea. Remember that we proved the lemma that if the left guys are injective, then there is no stackification going on, so $\tilde{\mathcal{C}} = \mathcal{C}$.

Here's the idea. If $k \in K_1^0$ is a section.

$$F_1(k) \xrightarrow{H} F_2(k)$$

is the same thing as a section $h(k) \in K_2^{-1}$ such that $dh(k) = f_2(k) - f_1(k)$.

Now all of this is supposed to lead up to a motivation for th cotangent complex.

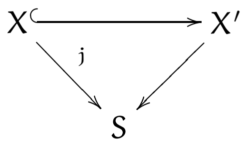
Preliminary definition. Let $f : X \to S$ be a morphism of schemes. The *truncated tangent complex*, denoted $\tau_{\leq 1}\mathbb{T}_{X/S}[1] \in \tilde{\mathcal{C}}^{[-1,0]}(|X|)$ is the complex with

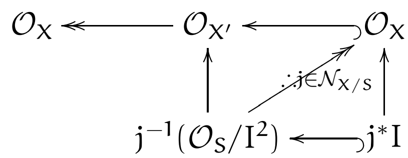
$$ch(\tau_{\leq 1}\mathbb{T}_{X/S}[1]) \xrightarrow{\sim} \underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$$

Problems with this:
a) This doesn't see $\mathcal{O}_X$-module structure.
b) This is not the full complex. This isn't really a problem.

So what is this complex? Let's try to figure it out.

Proposition. Let $j : X \hookrightarrow S$ be a closed immersion defined by an ideal I. Then $\tau_{\leq 1}\mathbb{T}[1]$ is quasiisomorphic to $\mathcal{N}_{X/S}[1]$ where $\mathcal{N}_{S/S} := \underline{\mathrm{Hom}}(j^*I, \mathcal{O}_X)$.

Proof. Translate  this into more sheaf-theoretic language:

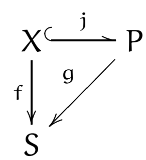


∎

Proposition. Let $f : X \to S$ be a smooth morphism. Then $\tau_{\leq 1}\mathbb{T}_{X/S}[1] \cong T_{X/S}[1] = (T_{X/S} \to 0)$.

Proof. We already know that $\mathcal{H}^0(\tau_{\leq 1}\mathbb{T}_{X/S}[1]) = 0$. Hence this complex is quasiisomorphic to $\mathcal{H}^1(\tau_{\leq 1}\mathbb{T}_{X/S}[1]) = T_{X/S}$. ∎

Proposition. Suppose we are given a commutative diagram



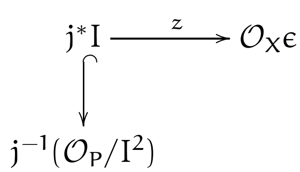
where g is smooth and j is an immersion. Then

$$\tau_{\leq 1}\mathbb{T}_{X/S}[1] \cong (j^*T_{P/S} \to \mathcal{N}_{X/P}).$$

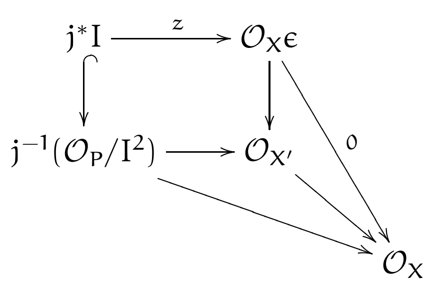
If I is the ideal of X in P, then this is dual to $I/I^2 \xrightarrow{d} j^*\Omega^1_{P/S}$.

Proof. $z : j^*I \to \mathcal{O}_X$ (section of $\mathcal{N}_{X/P}$), I have to produce for you an extension of x.

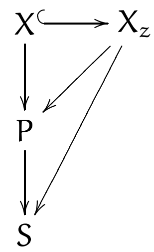
Consider this diagram



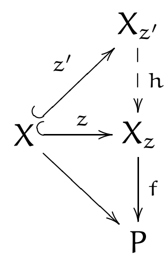
Construct the pushout



which gives us



We then get a noncommutative (!) diagram:



where the composition of the two vertical maps is called f'.

Then for $z, z' \in \mathcal{N}_{X/P}$. Then we get $(f \circ h - f') \in j^* T_{P/S}$.

The upshot is that there is a fully faithful functor

$$\mathrm{pch}(j^* T_{P/S} \to \mathcal{N}_{X/P}) \to \underline{\mathrm{Exal}}_S(X, \mathcal{O}_X).$$

Claim. The induced morphism of stacks

$$\mathrm{ch}(j^* T_{P/S} \to \mathcal{N}_{X/P}) \to \underline{\mathrm{Exal}}_S(X, \mathcal{O}_X)$$

is an equivalence.

(The proof was sketched in words.)

Now we have some problems.

a) choice of factorization of f (not a big deal — you can show you always get a quasi-isomorphic result)

b) factorization doesn't always exist (see Ravi's webpage for examples).

Replacement for factorization.

Suppose we have $f : X \to S$. Then we have a forgetful functor F from (sheaves of $f^{-1}(\mathcal{O}_s)$-algebras) to (sheaves of sets). Then F has a left-adjoint $\Omega \mapsto f^* \mathcal{O}_X\{\Omega\}$.

What's the idea? Well, if $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$. Well, we just choose a bunch of elements of A such that

$$\begin{array}{ccc} B[x_i] & \xrightarrow{x_i \mapsto f_i} & A \\ \uparrow & \nearrow & \\ A & & \end{array}$$

Do the analogous thing for sheaves.

Definition. The *truncated tangent complex* of $\mathcal{O}_X$-modules of f is the complex

$$\underline{\mathrm{Hom}}(\Omega^1_{f^{-1}\mathcal{O}_S\{F(\mathcal{O}_X)\}/f^{-1}\mathcal{O}_S}, \mathcal{O}_X) \to \underline{\mathrm{Hom}}(I/I^2, \mathcal{O}_X).$$

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