DEFORMATION THEORY WORKSHOP: OLSSON 6

ROUGH NOTES BY RAVI VAKIL

Recall that we are talking about Picard stacks. I'll remind you of the most salient poitns. T is a topological space. We have $(\mathcal{P}, +, \sigma, \tau)$. We don't have commutativity and associtivity of $+$ "on the nose", and σ and τ keep track of the isomorphisms that used to be equalities for usual addition.

Today, we'll consider a two-term complex $K^\bullet \in C^{[-1,0]}(T)$, where that superscript $[-1, 0]$ corresponds to the fact that our two terms are considered to be in gradings -1 and 0, i.e. our two-term complex is $K^{-1} \to K^0$.

We define $\mathrm{pch}(K^\bullet)$ ("pre-stack = pre-champs") as follows.

$\mathrm{pch}(K^\bullet)_U$ is a category. Objects are $x \in K^0(U)$, and morphisms $x \to y$ is an element $z \in K^{-1}(U)$ such that $dz = y - x$.

You can "stackify" this to get $\mathrm{ch}(K^\bullet)$.

If $\mathcal{P}$ is a Picard stack, then $\mathrm{HOM}(\mathrm{ch}(K^\bullet), \mathcal{P}) \to \mathrm{HOM}(\mathrm{pch}(K^\bullet), \mathcal{P})$ is an isomorphism. (HOM refers to the *category* of homomorphisms.)

Remark. $\mathrm{pch}(K^\bullet) \to \mathrm{ch}(K^\bullet)$ is fully faithful: stackifying just introduces new objects and morphisms.

Remarks. i) $f : K_1^\bullet \to K_2^\bullet$ induces a moprhism of Picard stacks $\mathrm{ch}(f) : \mathrm{ch}(K_1) \to \mathrm{ch}(K_2)$.

ii) Suppose $f_1, f_2 : K_1^\bullet \to K_2^\bullet$ and a homotopy h between f_1 and f_2.

$h : K_1^0 \to K_2^{-1}$ such that for all $x \in K^0$, $f_1(x) - f_2(x) = dh(x)$ and $f_1^{-1} - f_2^{-1} = hd$.

Then we get an isomorphism of morphisms $\mathrm{ch}(h) : \mathrm{ch}(f_1) \to \mathrm{ch}(f_2)$, i.e. for all $x \in \mathrm{pch}(K_1)$, we get an isomorphism $\mathrm{ch}(f_1)(x) \to \mathrm{ch}(f_2)(x)$.

$x \in K_1^0$, there exists $z \in K_2^{-1}$ such that $dz = f_2(x) - f_1(x)$. (That needs to be patched slightly.)

Lemma. If K^{-1} is flasque, then $\mathrm{pch}(K^\bullet)$ is a stack (i.e. it doesn't need to be stackified).

Proof. $\pi : \mathrm{pch}(K^\bullet) \to \mathrm{ch}(K^\bullet)$.

Let $U \subset T$ open and $x \in \mathrm{ch}(K^\bullet)_U$.

Date: Tuesday July 31, 2007.

Let $\mathcal{L}$ be the sheaf on U which to any open V associates the set of pairs $(y,,l), y \in K^0(V)$, and

$$l : \pi(y) \to x|_V$$

in $ch(K^\bullet)_V$.

Claim. $\mathcal{L}$ is a $K^{-1}|_U$-torsor.

Remark. $(y', l') \in \mathcal{L}$. $\pi(y) \xrightarrow{l} x|_V \xrightarrow{l'^{-1}} \pi(y')$. Now my functor is fully faithful, so this comes from a unique element in my prestack, so this gives some element $z \in K^{-1}$.

$\mathcal{L}$ is classified by an element $[\mathcal{L}] \in H^1(U, K^{-1}|_U) = 0$. Question: why does $\mathcal{L}$ locally have a section?

Observations. a) The sheaf associated to the presheaf

$$U \mapsto \text{the set of isomorphism classes of } ch(K^\bullet)_U$$

The answer is $\mathcal{H}^0(K^\bullet) := K^0/\operatorname{im}(K^{-1} \to K^0)$.

b) What is the automorphism group of an object $x \in ch(K^\bullet)_U$? Answer: $\mathcal{H}^{-1}(K^\bullet) = \ker(K^{-1} rightarrow K^0)$.

Idea: Consider the pre-stack. $x \in K^0(U)$, $\operatorname{Aut}(x) = \{z \in K^{-1}(U) : dz = x - x = 0\}$.

Corollary. If $f : K_1^\bullet \to K_2^\bullet$ is a quasi-isomorphism, then $ch(f) : ch(K_1^\bullet) \to ch(K_2^\bullet)$ is an equivalence of categories.

There is something to check here of course.

Let $\tilde{C}^{[-1,0]}(T) \subset C^{[-1,0]}(T)$ be the full subcateogry of complexes $K^{-1} \to K^0$ with K^{-1} injective. (This is nontraditional notation.)

Theorem. This ch constructions induces an equivalence of 2-categories $\tilde{C}^{[-1,0]}(T) \to ($ Picard stacks over $T)$.

You certainly shouldn't know what an equivalence of 2-categories is, but in the course of the proof, when we say what we need to prove, it will become clear.

Lemma. Suppose $f : \mathcal{X} \to \mathcal{Y}$ is a morphism of stacks, and $\bar{f} : X \to Y$ is the corresponding map on sheaves of isomorphism classes. Assume $\bar{f}$ is an equivalence and for all $U \subset T$ and $x \in \mathcal{X}_U$ the map of sheaves $\operatorname{Aut}_{\mathcal{X}}(x) \to \operatorname{Aut}_{\mathcal{Y}}(f(x))$ is an isomoprhism then f is an isomorphism.

Sketch of proof (that you should feel free to ignore). Given $x, y \in \mathcal{X}_U$ we want

$$\underline{\operatorname{Isom}}_{\mathcal{X}}(x, y) \to \underline{\operatorname{Isom}}_{\mathcal{Y}}(f(x), f(y))$$

to be an isomorphism. We may as well show that the map on sheaves is an isomorphism. So we'll need to show injectivity and surjectivity.

First, injectivity: given $\alpha, \beta : x \to y$. $f(\alpha) = f(\beta)$, $f(x) \to f(y)$. $\alpha^{-1} \circ \beta \in \ker(\mathrm{Aut}_{\mathcal{X}}(x) \to \mathrm{Aut}_{\mathcal{Y}}(f(x)))$. This implies $\alpha = \beta$.

Next we deal with surjectivity. This is a bit trickier, and we'll use injectivity. $\sigma : f(x) \to f(y)$. By injectivity, it is enough to show that σ is in the image locally. Now x, y map to the same thing in X. So locally there exists $\tau : x \to y$, $\sigma^{-1} \circ f(\tau) : f(x) \to f(x)$.

Essential surjectivity: $y \in \mathcal{Y}_T$. There exists a covering $T = \cup_i \mathcal{U}_i$ and (x_i, l_i) with $x_i \in \mathcal{X}_{U_i}$, $l_i : f(x_i) \xrightarrow{\sim} y|_{U_i}$ in $\mathcal{Y}_{U_i}$. Then on U_{ij}, there exists a *unique* isomorphism $\sigma_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ such that

$$\begin{array}{ccc} f(x_i)|_{U_{ij}} & \xrightarrow{f(\sigma_{ij})} & f(x_j)|_{U_{ij}} \\ \downarrow l_i & \swarrow l_j & \\ y|_{U_{ij}} & & \end{array}$$

This ends the sketch of the proof of the lemma.

Let's now try to prove that theorem about this equivalence of 2-categories.

But first, a philosophical remark: why am I doing all this? Most of our deformation problems come to us as Picard stacks. That means that there is some two-term complex around. That's the idea.

Lemma. Suppose $\mathcal{P}$ is a Picard stack over T, and $\{U_i\}$ is a collection of open subsets, $k_i \in \mathcal{P}(U_i)$ for all i. Let me define $K = \oplus_i \mathbb{Z}_{U_i}$. (Recall that $\mathbb{Z}_{U_i}$ is the extension by 0 of the constant sheaf on U_i, i.e. $j_! \mathbb{Z}$ where $j : U \hookrightarrow T$.) Then there is a morphism $F : \mathrm{ch}(0 \to K) \to \mathcal{P}$ and isomorphisms $\sigma_i : F(1 \in \mathbb{Z}_{U_i}(U_i)) \to k_i$ and any choice is unique up to unique isomorphism.

Example. $\mathrm{Pic}(X) = \mathrm{ch}(\mathcal{O}_X^\times \to 0)$.

Proof. Choose data
a) $\{U_i \subset T\}_{i \in I}$,
b) for all i, $k_i \in \mathcal{P}(U_i)$, $K^0 = \oplus_i \mathbb{Z}_{U_i}$
such that for all $V \subset$, $k \in \mathcal{P}_V$, there exists a covering $V = \cup V_j$ such that$k|_{V_j} \cong k_i$ some i wit $V_j \subset U_i$.

$F : \mathrm{ch}(0 \to K^0) \to \mathcal{P}$.

$K^{-1}(V) = \{(x, l), x \in K^0(V), l : F(0) \xrightarrow{\sim} F(x)\}$, $K^{-1} \to K^0$ given by $(x, l) \mapsto x$.

Let's figure out addition.

$$\begin{array}{ccccc} F(0) & \xrightarrow{\sim} & F(0) + F(0) & \xrightarrow{l+l'} & F(x) + F(x') \\ & & & \overset{?}{\searrow} & \downarrow \cong \\ & & & & F(x + x') \end{array}$$

The rest I'll leave for you to think through.

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