DEFORMATION THEORY WORKSHOP: OLSSON 5

ROUGH NOTES BY RAVI VAKIL

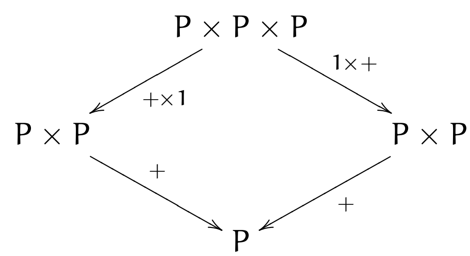
This wek I want to give you some sense of the cotangent complex.

In each of our examples, we have some commonalities floating around: (first order) automorphisms are in H^0, first-order deformations are in H^1, obstructions are in H^2. This is a clue that there is some complex lying around.

We also want to make precise something we've been saying informally as "morphisms of deformation problems".

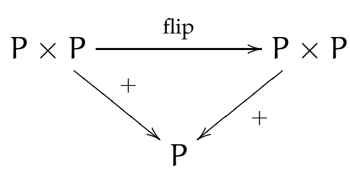
We start by defining the notion of a Picard category. (A better word would be **abelian group category**. A **Picard category** is a groupoid P together with the following extra structure.

(a) A functor $+ : P \times P \to P$
(b) We're looking at a groupoid, so we don't want associativity or commmutatity on the nose, so we only want these to hold "up to isomorphism". So we want an isomorphism of functors from the left side to the right side of this diagram.



$\sigma_{x,y,z} : (x + y) + z \xrightarrow{\sim} x + (y + z)$

(c) a natural transformation of functors from the lower left to the composition of the other two morphisms in



$\tau_{x,y} : x + y \xrightarrow{\sim} Y + X$

They must staisfy:

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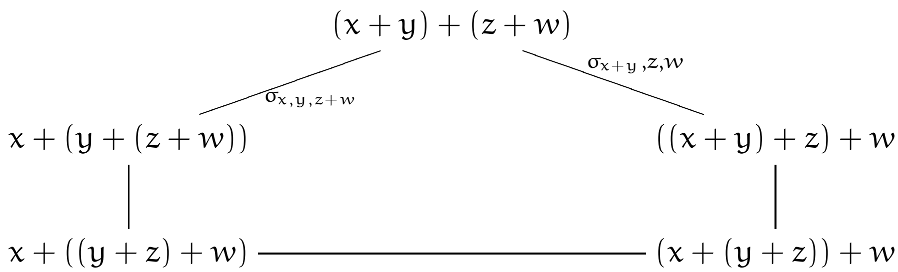
(0) for all $x \in P$, the functor $P \to P$, $y \mapsto x + y$ is an equivalence.

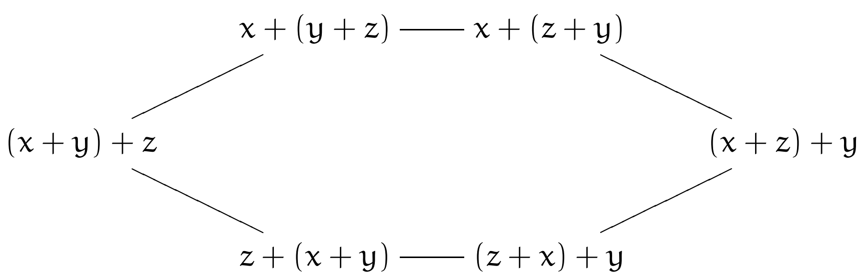
(i) (Pentagon axiom) The following diagram commutes (name the arrows appropriately)



(ii) $\tau_{x,x} = id$ for all $x \in P$.

(iii) for all $x, y \in P$, $\tau_{x,y} \circ \tau_{y,x} = id$.

(iv) (hexagon axiom) The following diagram commutes (name the arrows appropriately):

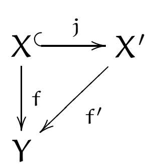


Example. X scheme, $\mathrm{Pic}(X)$ groupoid of line bundles on X.

$$\otimes : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathrm{Pic}(X).$$

Now we'll come to the example which most interests us, in deformation theory. This is the "primordial deformation problem", from which all others arise.

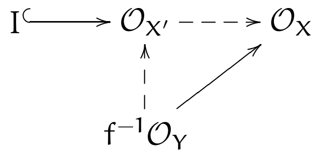
Example. Take a morphism $f : X \to Y$ of schemes, and let I be a quasicoherent $\mathcal{O}_X$-module. Define an I-*extension of* X *over* Y as a diagram



where j is a square-zero closed immersion, together with an isomorphism $\iota : I \to \ker(\mathcal{O}_{X'} \to \mathcal{O}_X)$. As usual, that kernel is a priori an $\mathcal{O}_{X'}$ module, but it is an $\mathcal{O}_X$-module.

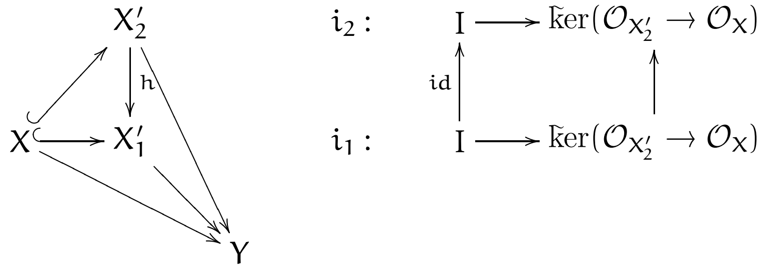
Let $\mathbf{Exal}_Y(X, I)$ be the category of I-extensions of X.

Remarks. (a) "Exal" is historical notation. It stands for "Extensions of algebras" of $\mathcal{O}_X$ by I. The following diagram may make that clearer.

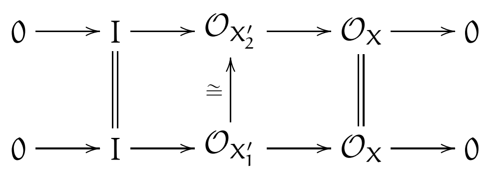


(b) If $A \to B$ is a morphism of sheaves of algebras on a topological space (or even site) T and I is a B-module, then I get a category $\mathbf{Exal}_A(B, I)$. This was studied first in the case when the space was a point, by Quillen. In fact, if you get confused in what follows, you should just deal with that case, when the space is a point. You won't lose too much.

Remark. $\mathbf{Exal}_Y(X, I)$ is a groupoid. The following diagram might help.



with



(c) If $U \subset X$, then there is a restriction functor

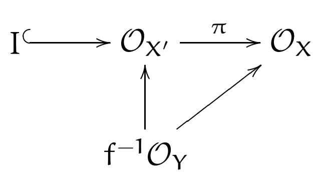
$$\mathbf{Exal}_Y(X, I) \to \mathbf{Exal}_Y(U, I|_U).$$

Remark. $u : I \to J$ is a map of $\mathcal{O}_X$-modules. Then there is a functor

$$u_* : \mathbf{Exal}_Y(X, I) \longrightarrow \mathbf{Exal}_Y(X, J)$$

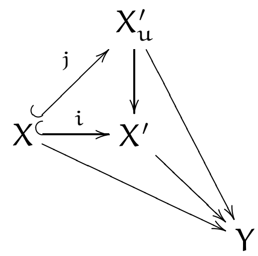
$$X' \longmapsto X'_u$$

defined by



where $\mathcal{O}_{X'_u}$ is defined by

$$\mathcal{O}_{X'_u} = \mathcal{O}_{X'} \oplus_I J = (\mathcal{O}_{X'}[J]/\{(i, -u(i)) \mid i \in I\}$$



We next state a lemma that Brian proved (or at least stated) before.

Lemma. If I and J are two quasicoherent $\mathcal{O}_X$-modules, then

$$(pr_{1*}, pr_{2*}) : \mathbf{Exal}_Y(X, I \oplus J) \to \mathbf{Exal}_Y(X, I) \times \mathbf{Exal}_Y(X, J)$$

is an equivalence of categories.

Brian comments: This is like condition (H2), but in a more general setting. In some sense it is a little easier, because it is a pure algebra problem.

We now define the "sum functor". Let $\Sigma : I \oplus I \to I$ be the summation map. Then we get

$$\begin{array}{ccc} \mathbf{Exal}_Y(X, I) \times \mathbf{Exal}_Y(X, I) & \xrightarrow{\sim} & \mathbf{Exal}_Y(X, I \oplus J) \\ & \searrow^{+} & \downarrow{\Sigma_*} \\ & & \mathbf{Exal}_Y(X, I) \end{array}$$

Then you have to write out σ and τ, and check that all desired diagrams commute. That is left for you to work out; it is frankly no fun.

But it is of a similar flavor from lecture (approximately) 2, when I used it to give a vector space structure on the tangent space.

Now I'll give an example of a Picard category that will look rather trivial. But the main theorem tomorrow will say that this is essentially all examples. We'll use better notation tomorrow.

Example of a Picard category. Let $f : A \to B$ be a homomorphism of abelian groups. Define the category P_f as follows. The objects are elements of B, and a morphism $x \to y$ is an element $h \in A$ with $f(h) + x = y$. To describe the additive structure, we need to define a functor $+ : P_f \times P_f \to P_f$. On the level of objects, this is clear: we add in g. For morphisms:

$$\begin{array}{ccc} (x, y) & \longmapsto & x + y \\ \downarrow{\scriptstyle (h,g)} & & \downarrow{\scriptstyle (h,g)} \\ (x', y') & \longmapsto & x' + y' \end{array}$$

We check:

$$f(h + g) = f(h) + f(g) = (x' - x) + (y' - y) = (x' + y') - (x + y).$$

In this case the σ and τ are actual equalities.

Now let's jazz this example up.

Suppose T is a topological space (or a site). A **Picard (pre-)stack** over T is a (pre-)stack $\mathcal{P}$ with morphisms of stacks $(+, \sigma, \tau)$ such that for all $U \subset T$ the fiber $(P(U), +, \sigma, \tau)$ is a Picard category.

Here is an example of that.

Example. $\mathrm{Pic}(\cdot)$ defines a Picard stack on the topological space $|X|$.

Example. $\mathbf{Exal}_Y(\cdot, I)$ gives a Picard stack. on the topological space $|X|$.

Example. $f : A \to B$ is a homomorphism of sheaves on a topological space T then we get a Picard pre-stack which will be very important for us, which we denote $\mathrm{pch}(A \to B)$. (The notation pch is from the french: pre-champs = pre-stack.)

Let's get a few more definitions down, which you'll need for the homework.

Now this should be a generalization of abelian groups, so we should be able to do things such as kernels, cokernels, hom's, etc.

Suppose T is a topological space, and $\mathcal{P}_1$ and $\mathcal{P}_2$ are Picard stacks over T. Then a morphism is $\mathcal{P}_1 \to \mathcal{P}_2$ is a pair (F, ι) where $F : \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of stacks, and $\iota : F(x+y) \xrightarrow{\sim} F(x) + F(y)$ such that

$$\begin{array}{ccc} F(x+y) & \xrightarrow{\iota} & F(x)+F(y) \\ \downarrow{\scriptstyle\tau} & & \downarrow{\scriptstyle\tau} \\ F(y+x) & \xrightarrow{\iota} & F(y)+F(x) \end{array}$$

commutes, and

$$\begin{array}{ccccc} F((x+y)+z) & \xrightarrow{\iota} & F(x+y)+F(z) & \xrightarrow{\iota} & (F(x)+F(y))+F(z) \\ \downarrow{\scriptstyle F(\sigma)} & & & & \downarrow{\scriptstyle\sigma} \\ F(x+(y+z)) & \xrightarrow{\iota} & F(x)+F(y+z) & \xrightarrow{\iota} & F(x)+(F(y)+F(z)) \end{array}$$

commutes.

Tomorrow, we'll define Hom, the identity element, kernels, and $\otimes$.

E-mail address: vakil@math.stanford.edu