DEFORMATION THEORY WORKSHOP: OLSSON 4

ROUGH NOTES BY RAVI VAKIL

Last time, I introduced the abstraction notion of obstruction theory, and today I'd like to do some more examples.

Let's consider deformings embeddings into, for example, projective space.

Suppose $A' \to A$ is a surjective map of rings with square-zero kernel J, and

$$\begin{array}{c} P' \\ \downarrow \\ \operatorname{Spec} A' \end{array}$$

a smooth scheme with reduction

$$\begin{array}{c} P \\ \downarrow \\ \operatorname{Spec} A \end{array}$$

and also

$$\begin{array}{ccc} X & \overset{j}{\hookrightarrow} & P \\ & {\scriptstyle\text{smooth}}\searrow & \downarrow \\ & & \operatorname{Spec} A \end{array}$$

Here you can think about P as projective space over A, and similarly P' over A' (i.e. $P = \mathbb{P}^n_A$, $P' = \mathbb{P}^n_{A'}$.

The first problem for today is:

Problem. How can we lift this whole diagram to a diagram

$$\begin{array}{ccc} X' & \overset{j}{\hookrightarrow} & P' \\ & {\scriptstyle\text{smooth}}\searrow & \downarrow \\ & & \operatorname{Spec} A' \end{array}$$

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We have a diagram of sheaves of algebras on the topological space $|X|$:

$$\begin{tikzcd} & \mathcal{O}_X \\ j^{-1}\mathcal{O}_{P'} \arrow[r, two heads] & j^{-1}\mathcal{O}_P \arrow[u] \\ A' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

that we want to complete to

$$\begin{tikzcd} \mathcal{O}_{X'} \arrow[r, dashed] & \mathcal{O}_X \\ j^{-1}\mathcal{O}_{P'} \arrow[u, dashed] \arrow[r, two heads] & j^{-1}\mathcal{O}_P \arrow[u] \\ A' \arrow[u] \arrow[r] & A \arrow[u] \end{tikzcd}$$

We could cut things up into affines as we did yesterday, but that's no fun. So we'll do something slightly different.

Let $\mathcal{L}$ be the sheaf on $|X|$ which to any open $U \subset X$ associates the set of diagrams

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, hook, "j"] & U' \arrow[d, hook, "j'"] \arrow[ddr, "\text{smooth}"] & \\ P \arrow[r, hook] \arrow[dr] & P' \arrow[dr] & \\ & \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A' \end{tikzcd}$$

You can check that this is a sheaf. The key fact is that (as in Max's talk earlier today, see "Lieblich 4") there are no nontrivial automorphisms of the objects.

Now what are the global sections of $\mathcal{L}$?

We've seen that If U is affine, then U' is uniquely determined, so there is no information in U', only in the map j. So let's ask ourselves how many ways can we fill in the dashed arrow in the following:

$$\begin{tikzcd} U \arrow[r, hook] \arrow[d, "j"] & U' \arrow[d, dashed] \\ P \arrow[r, hook] \arrow[d] & P' \arrow[d] \\ \operatorname{Spec} A \arrow[r, hook] & \operatorname{Spec} A' \end{tikzcd}$$

The set of arrows filling in the diagram form a torsor under

$$\mathrm{Hom}((i \circ j)^* \Omega^1_{P'/A'}, J \otimes \mathcal{O}_U)$$

This is the universal property of differentials.

$$= j^* T_{P/A} \otimes_A J.$$

There is an action onf $j^* T_{P/A} \otimes J$ on $\mathcal{L}$.

Now we have the conormal bundle $j^* \mathcal{I} = \mathcal{I}/\mathcal{I}^2$, where $\mathcal{I} \subset \mathcal{O}_P$ is the ideal sheaf of X.

We have the nonlinear differential map $\mathcal{I} \xrightarrow{d} j^* \Omega^1_{P/A}$ which vanishes on $\mathcal{I}^2$; modding the left by $\mathcal{I}^2$ turns thiis into a linear map, and this gives an exact sequence (which you can find in Hartshorne II.8):

$$0 \longrightarrow \mathcal{I}/\mathcal{I}^2 \xrightarrow{d} j^* \Omega^1_{P/A} \longrightarrow \Omega^1_{X/A} \longrightarrow 0$$

which we dualize to get

$$0 \longrightarrow T_{X/A} \xrightarrow{d} j^* T_{P/A} \longrightarrow \mathcal{N} \longrightarrow 0$$

where $\mathcal{N}$ is the *normal* bundle.

We tensor this with J, which preserves exactness because $\mathcal{N}$ is locally free (flat suffices):

$$0 \longrightarrow T_{X/A} \otimes J \xrightarrow{d} j^* T_{P/A} \otimes J \longrightarrow \mathcal{N} \otimes J \longrightarrow 0$$

Claim. $T_{X/A} \otimes J$ acts trivially on $\mathcal{L}$, and a sectino $\partial \in T_{X/A} \otimes J(U)$ corresponds to a diagram

$$\begin{array}{ccc}
 & \nearrow & U \\
 & & \downarrow \partial \\
U & \hookrightarrow & U' \\
\downarrow & & \downarrow \\
P & \hookrightarrow & P' \\
\downarrow & & \downarrow \\
\operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} A'
\end{array}$$

So we get an action of $\mathcal{N} \otimes J$ on $\mathcal{L}$

Proposition. $\mathcal{L}$ is a torsor under $\mathcal{N} \otimes J$.

I should first say what it means for a sheaf of sets to be a torsor under a torsor of a sheaf of groups.

Definition. (a) For all $U \subset X$, there exists a covering $\mathcal{U} = \cup U_i$ such that $\mathcal{L}(U_i) \neq \varnothing$.
(b) For every $U \subset X$, either $\mathcal{L}(U) \neq \emptyset$, or the action of $\mathcal{N} \otimes J(U)$ on $\mathcal{L}(U)$ is simply transitive.

Sketch of proof. Check that if U is affine the aciton of $\mathcal{N} \otimes J(U)$ on $\mathcal{L}(U)$ is simply transitive. That turns out to be precisely this exact sequence.

$$0 \to T_{X/A} \otimes J(U) \to j^*T_{P/A} \otimes J(U) \to \mathcal{N} \otimes J(V) \to 0.$$

$\square$

General fact: If G is a sheaf of abelian groups, then the set of isomorphism classes of G-torsors on $|X|$ are in canonical bijection with $H^1(X, G)$.

You may have already seen this before in the special case where $G = \mathcal{O}_X^*$ in a Hartshorne exercise, where you check that isomorphism classes of line bundles on X correspond to elements of $H^1(X, \mathcal{O}_X^*)$.

In particular, $\mathcal{L} \leftrightarrow [\mathcal{L}] \in H^1(X, \mathcal{N} \otimes J)$.

In our case, choose a covering of $X = \cup_i U_i$ with U_i affine and $s_i \in \mathcal{L}(U_i)$. The trouble is, they might not agree. On $U_i \cap U_j$, we get two sections $s_i|_{U_{ij}}$ and $s_j|_{U_{ij}} \in \mathcal{L}(U_{ij})$. There's no reason for them to be equal. But the (complicated-to-describe!) action of $\mathcal{N} \otimes J(U_{ij})$ on $\mathcal{L}(U_{ij})$ is simply transitive, which implies that there is a unique $x_{ij} \in \mathcal{N} \otimes J(U_{ij})$ such that $x_{ij} * s_i|_{U_{ij}} = s_i|_{U_{ij}}$.

Exercise. Check that $\{x_{ij}\}$ is a Cech 1-cocycle, so we get a class in $H^1(X, \mathcal{N} \otimes J)$.

Now $\mathcal{L}$ trivial $\leftrightarrow \mathcal{L}(X) \neq 0 \leftrightarrow [\mathcal{L}] \in H^1(X, \mathcal{N} \otimes J)$ is zero.

Summary. Consider the diagram

$$\begin{array}{ccc} X & & \\ \downarrow & & \\ P & \hookrightarrow & P' \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} A' \end{array}$$

(i) There exists a canonical obstruction. $o(j) \in H^1(X, \mathcal{N} \otimes J$ whose vanishing is necessary and sufficient for existence of a lifting of j.

(ii) The set of liftings j' of j form a torsor over $H^0(X, \mathcal{N} \otimes J)$ if $o(j) = 0$.

Remark. $0 \to T_{X/A} \to j^*T_{P/A} \to \mathcal{N} \to 0$ induces the long exact sequence

$$H^0(X, \mathcal{N} \otimes J) \longrightarrow H^1(X, T_{X/A} \otimes J) \longrightarrow H^1(X, j^*T_{P/A} \otimes J) \longrightarrow$$

$$H^1(X, \mathcal{N} \otimes J) \xrightarrow{\partial} H^2(X, T_{X/A} \otimes J)$$

What is $\partial(o(j))$? Answer: $o(g)$.

Example. Suppose P is a smooth proper surface over k, and $X \subset P$ is a smooth rational curve with $X \cdot X = 1$. Then Hartshorne V.1.4.1 will tell you that $\deg \mathcal{N} = -1$. Hence $H^1(X, \mathcal{N} \otimes J) = 0$, $H^0(X, \mathcal{N} \otimes J) = 0$.

If I deform P, can I deform X? In how many ways? This tells you that you can deform the curve, and deform it uniquely.

You can use it to show quickly that there are 27 lines on any smooth cubic surface (over an algebraically closed field).

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