DEFORMATION THEORY WORKSHOP: OLSSON 3

ROUGH NOTES BY RAVI VAKIL

Today's topic is **Obstruction theories**. I want to start with yesterday's example. I'd like to start by summarizing what I said yesterday, hopefully a little better.

Let me start with $\pi : A' \to A$, a surjection of rings, $I = \ker(\pi)$ a square zero ideal ($I^2 = 0$). We'll think of it as an A-module. It is a priori an A'-module, but it is annnihilated by I.

Let $g : X \to \operatorname{Spec} A$ be a smooth separated scheme. (Again, separated is not necessary.)

The problem is to understand the liftings over A', by which we mean a Cartesian diagram

$$\begin{array}{ccc} X & \hookrightarrow & X \\ \downarrow g & & \downarrow f \text{ smooth} \\ \operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} A' \end{array}$$

We defined the *deformation functor* $\operatorname{Def}_X : \mathbf{Alg}/A \to \mathbf{Set}$ given by

$$(f : C \to A) \mapsto \left\{ \begin{array}{ccc} X & \longrightarrow & X_C \\ \downarrow & & \downarrow \\ \operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} C \end{array} \right\} / \cong$$

Yesterday, we saw that $T_{\operatorname{Def}_X} = H^1(X, T_X)$.

When X is affine, we saw:

(1) there exists a lifting $X' \to \operatorname{Spec} A'$, and

(2) any twoo liftings are isomorphic

(3) the group of automorphisms of any lifting $X' \to \operatorname{Spec} A.$ is canonically isomorphic to $H^0(X, T_X \otimes I)$.

(For (1): this is easy if $A' = A[\epsilon]$, but uses smoothness in other situations, e.g. $A = \mathbb{F}_p$, $A' = \mathbb{Z}/p^2$. For (2) we used the infinitesimal lifting property. (3) turned out to be essentially the universal property of the tangent bundle, or better, the cotangent bundle/sheaf.)

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Now for general X, if $X' \to \operatorname{Spec} A'$ is a smoth lifting, I get a bijection

$$\phi_{X'} \operatorname{Def}_X(A' \to A) \to H^1(X, T_X \otimes I).$$

(Yesterday the fixed lifting was $X[I] \to \operatorname{Spec} A[I]$.) Here's the definition of $\phi_{X'}$. Cover $X = \cup U_i$ where U_i are affine.

$$X'' \in \operatorname{Def}_X(A')$$

. For all i,

$$\begin{array}{ccc} & & U_i'' \\ & & \downarrow{\scriptstyle \sigma_i} \\ U & & \\ & & U_i' \end{array}$$

choose $\sigma_i : U_i'' \to U_i'$ for all i. This gives for all i and j,

$$U_{ij}' \xrightarrow{\sigma_j^{-1}} U_{ij}'' \xrightarrow{\sigma_i} U_{ij}'$$

where the composition is $x_{ij} \in H^0(U_{ij}, T_X \otimes I)$.

Another way to say it (for experts): given X' and x'', we get a sheaf

$$\underline{\operatorname{Isom}}(X', X'')$$

on $|X|$, given by assigning to $(U \subset X)$ the following:

$$\begin{array}{ccc} & \nearrow & U'' \\ U & & \downarrow \\ & \searrow & U' \end{array}$$

This is a torsor under $T_X \otimes I$.

Then there is a general fact that you probably know for line bundles, which are classified by $H^1(\mathcal{O}^*)$: torsors are always classified by H^1 of whatever the sheaf of groups is.

So in this case, we get $H^1(X, T_X \otimes I)$.

Then we could define $\phi_{X'}$ by $x'' \mapsto [\{x_{ij}\}] = \underline{\operatorname{Isom}}(X', X'')$.

Question: When does there exist $X' \to \operatorname{Spec} A'$?

Let's try to build one.

Let $\mathcal{U} = \{U_i\}$ be a covering of X by affines. We know how to lift each U_i; this can be done in only one way (up to *nonunique* isomorphism in general!). Fix liftings $U_i' \to \operatorname{Spec} A'$.

For all i and j, choose an isomorphism

$$\phi_{ji} : U_i'|_{U_{ij}} \to U_j'|_{U_{ij}}.$$

We'll want to choose these judiciously, so that everything can come together to get some X'. For example, we'll want these to "agree on triple overlaps". More precisely,

$$\begin{array}{ccc} U_i'|_{U_{ijk}} & \xrightarrow{\phi_{ki}} & U_k'|_{U_{ijk}} \\ & {\searrow^{\phi_{ji}}} \quad {\nearrow_{\phi_{kj}}} & \\ & U_j'|_{U_{ijk}} & \end{array}$$

should commute.

Define ∂_{jk}, an automorphism of U_i', by $\phi_{ki}^{-1} \circ (\phi kj \circ \phi_{ji})$. This lies in $H^0(U_{ijk}, T_X \otimes I)$ (as these classify automorphisms of U_i').

Now we have a lemma that we're not going to prove. (The diagram is no fun. We saw a similar proof dealing with a simpler diagram yesterday.)

Lemma.
(i) $\{\partial_{ijk}\}$ is a Cech 2-cocyle.
(ii) if ϕ_{jk}' is a second choice of isomorphisms, yielding some other $\{\partial_{ijk}'\}$, then the difference $\{\partial_{ijk} - \partial_{ijk}'\}$ is a Cech boundary.

Thus we get a well-defined cohomology class

$$o(g) \in H^2(X, T_X \otimes I).$$

[Might the notatation $\mathrm{ob}(g)$ be better?

Proposition. There exists a lifting $X' \xrightarrow{g'} \operatorname{Spec} A'$ of g if and only if $o(g) = 0$.

Idea of proof: try choosing the ϕ_{ji} at random. That may not work, but then $\{\partial_{ijk}\}$ that is a boundary. This means that you get elements of $H^0(U_{ijk}, T_X \otimes I)$. This tells you how to adjust your guess so that the obstruction is 0. Then you can build your X'.

Let me summarize. (We'll see this in even more pleasant detail next week, when we get a glimpse of the cotangent complex.)

Summary.

a) There exists an obstruction $o(g) \in H^2(X, T_X \otimes I)$ such that $o(g) = 0$ if and only if $\mathrm{Def}_X(A' \to A) \neq \varnothing$.

b) If $o(g) = 0$, then the set of isomorphism classes of liftings form a torsor under $H^1(X, T_X \otimes I)$.

c) For any lifting of g the group of uatomorphisms is cannonically isomorphic to $H^0(X, T_X \otimes I)$.

Formalizing the notion of an obstruction theory.

Let me now formalize (because it will come up later) the notion of an **obstruction theory**.

Suppose Λ is a ring, and let F be a functor

$$F : \Lambda\textbf{-Alg} \to \textbf{Set}.$$

Definition. An obstruction theory for F consists of the following data.

(i) For every surjective morphism $A \to A_0$ of Λ-algebras with kernel a nilpotent ideal and A_0 reduced and an $a \in F(A)$

$$\mathcal{O}_a : \textbf{finite type } A_0\textbf{-modules} \to \textbf{finite type } A_0\textbf{-modules}$$

In our example above, this would be taking an H^2.

(ii) For all diagrams $A' \twoheadrightarrow A \twoheadrightarrow A_0$ and $a \in F(A)$ where $A' \to A$ is surjective, $\ker(A' \to a) = J$ annihilated by $\ker(A' \to A_0)$ a class

$$o(a) \in \mathcal{O}_A(J)$$

which is zero if and only if a lifts to $F(A')$.

"This should be functorial in the natural way." It's worth guessing what this means. This will be included in the notes at some point.

Let's give one more example.

Example. Suppose X $\quad$ X' is a closed immersion defined by a square zero ideal J, and L is a line bundle on X.

Then the problem is to understand how/if we can lift our line bundle L to X'.

What do we mean by lifting? We mean the following: a pair (L', ι), L' a line bundle on X' and $\iota : j^*L' \cong L$. We say that $(L', \iota) \cong (L'', \iota'')$ if there exists an isomorphism $\sigma : L' \xrightarrow{\sim} L''$ such that they agree with the other maps, i.e. the following diagram commutes

$$\begin{array}{ccc} j^*L' & \xrightarrow{\sigma} & j^*L'' \\ & {\scriptstyle\iota}\searrow \quad \swarrow{\scriptstyle\iota]''} & \\ & L & \end{array}$$

We have an exact sequence of sheaves

$$0 \longrightarrow J \xrightarrow{g \mapsto 1+g} \mathcal{O}_{X'}^* \longrightarrow \mathcal{O}_X^* \longrightarrow 0$$

The map $g \mapsto 1 + g$ is an additive map: $f + g$ maps to $1 + f + g = (1 + f)(1 + g)$ so the image of $f + g$ is the sum of the images of f and g.

We take the long exact sequence:

$$0 \longrightarrow H^0(J) \longrightarrow H^0(\mathcal{O}^*_{X'}) \longrightarrow H^0(\mathcal{O}^*_X)$$

$$\longrightarrow H^1(J) \longrightarrow H^1(\mathcal{O}^*_{X'}) \longrightarrow H^1(\mathcal{O}^*_X)$$

$$\xrightarrow{\delta} H^2(J)$$

Then we can see:

Proposition.

a) There exists an obstruction $o(L) = \delta([L]) \in H^2(X, J)$ which is 0 if and only if there exists an (L', ι) lifting L.

b) Assume $H^0(X', \mathcal{O}^*_{X'}) \to H^0(X, \mathcal{O}^*_X)$ is surjective. Then if $o(L) = 0$, then the set of isomorphism clases of liftings (L', ι) is a torsor under $H^1(X, J)$.

c) For any lifting, the group of automorphisms is canonicaly in bijection with $H^0(X, J)$.

So we have two examples of obstruction theories: deformations of smooth schemes, and deformations of line bundles. In each case, we have three consecutive cohomology groups.

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