DEFORMATION THEORY WORKSHOP: OLSSON 2

ROUGH NOTES BY RAVI VAKIL

Without further ado, let's get back to where we are. Suppose we have $A \to R$, and the category A-**Alg**$/R$, the category of diagrams

$$\begin{array}{ccc} C & \xrightarrow{f} & R \\ \uparrow & \nearrow & \\ A & & \end{array}$$

and we have a functor $F : A\text{-}\mathbf{Alg}/R \to \mathbf{Set}$ a functor, and if for all $I, J \in \mathbf{Mod}_R$ the natural map

$$F(R[I \otimes J]) \to F(R[I]) \times F(R[J])$$

then we get a tagent space T_F. (In fact we got a little bit mroe: for all I, $F(R[I])$ is an R-module, and T_F is by definition $F(R[\epsilon])$.)

Let me remind you how we get the R-module structure. Sum is given by

$$+ : \quad F(R[\epsilon]) \times F(R[\epsilon]) \xrightarrow{\sim} F\ \mathrm{left}(R[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2, \epsilon_1\epsilon_2) \xrightarrow{\epsilon_i \mapsto \epsilon} F(R[\epsilon])$$

and multiplication is given by

$$\times f : F(R[\epsilon]) \to F(R[\epsilon])$$

induced by $R[\epsilon] \to R[\epsilon]$ given by $a + b\epsilon \mapsto a + fb\epsilon$. That's a ring homomorphism.

Problem 1. Suppose R is a ring, and $X \xrightarrow{g} \operatorname{Spec} R$ is separated and smooth. (Separatedness isn't necessary, but the assumption will simplify the exposition.)

Consider the function $\operatorname{Def}_X : \mathbf{Alg}/R \to \mathbf{Set}$. (Really we should write $\mathbb{Z}$-**Alg**$/R$.)

$\operatorname{Def}_X(C \xrightarrow{f} R)$ is the set of isomorphism classes of Cartesian diagrams

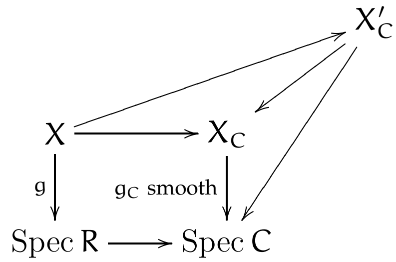
$$\begin{array}{ccc} X & \longrightarrow & X_C \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g_C \text{ smooth}} \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} C \end{array}$$

We will be interested in the case when $C \to R$ is a nilpotent closed immersion (also known as a nilpotent thickening).

Date: July 24, 2007.

We now have to say what we mean by "isomorphism classes". So we should say what a morphism of such diagrams is (i.e. what the category of such diagrams is). Then we'll know what isomorphisms are. A morphism of diagrams (from a diagram with X'_C in the upper right corner to the diagram with X_C in the upper right corner is a diagram an arrow

(1)



(Really I shold draw a commuting cube here.)

Remark. If $C = R[I]$ for some R-module I, then any morphism h as in (1) is an isomorphism. Here's why. In this case, X and X_C have the same underlying topological space. We have an exact sequence of sheaves on this topological space

(2)
$$\begin{array}{ccccccccc} 0 & \longrightarrow & I \otimes_R \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X_C} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \\ & & \| & & \downarrow h^* & & \| & & \\ 0 & \longrightarrow & I \otimes_R \mathcal{O}_X & \longrightarrow & \mathcal{O}_{X'_C} & \longrightarrow & \mathcal{O}_X & \longrightarrow & 0 \end{array}$$

Then h^* must be an isomorphism. We've only used flatness.

Proposition. For all $I, J \in \mathbf{Mod}_R$, $\mathrm{Def}_X(R[I \oplus J]) \to \mathrm{Def}_X(R[I]) \times \mathrm{Def}_X(R[J])$ is an isomorphism.

Brian Osserman will prove this tomorrow (see Osserman 3), so we will take it as given.

So how do we compute $T_{\mathrm{Def}\,X}$, or more generally the R-module $\mathrm{Def}_X(R[I])$?

There's a fancy version, and a hands-on version. I'll present the hands-on version now, and we may discuss the fancy version next week.

We begin with a very special case, when X is affine (and smooth).

We recall some facts, which are related to the background lectures on smoothness.

(1) $\mathrm{Def}_X(R[I])$ consists of one element.

(2) For any deformation (well, there's only one...)

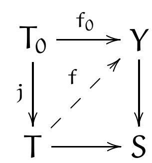
$$\begin{array}{ccc} X & \hookrightarrow & X' \\ \downarrow & & \downarrow \\ \mathrm{Spec}\, R & \longrightarrow & \mathrm{Spec}\, R[I] \end{array}$$

the set of maps $h : X' \to X'$ as in (1) is in canonical bijection with $H^0(X, T_X \otimes I)$.

Here is some discussion as to why these are true.

In fact $X[I] \to R[I]$ is a smooth lifting (as smoothness is preserved by base change). Here's why there is only one. If you have another X', then by the formal criterion for smoothness, we get maps $X' \to X[I]$ and $X[I] \to X'$. An argument like at (2) shows that they are isomorphic.

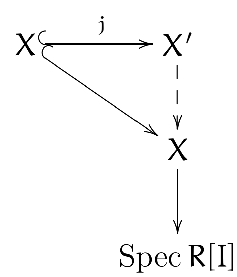
For the second fact, recall that



where j is a closed immersion by a square zero ideal J, then the set of arrows f filling in the diagram is a pseudo-torsor under $\mathrm{Hom}(f_0^*\Omega^1_{Y/S}, J)$. (Note: we are considering J as a sheaf on T_0, even though it is a sheaf on T_1. We can do this, as J is square 0.)

(Definition of pseudo-torsor: if the set is non-empty, then there is a group action that acts singly transitively on the set. In other words, the set is empty or a torsor.)

This recollection is possibly an exercise in Hartshorne, and depending on your definition, this is almost the definiton of Ω^1. It may have been in the background exercises.



We now understand our special case.

For a general $X \to \mathrm{Spec}\, R$ (i.e. not necessarily affine), this also shows that $(X[I] \to \mathrm{Spec}\, R[I]) \in \mathrm{Def}_X(R[I])$.

Choose a covering $X = \cup_i U_i$ with each U_i affine. Let $\mathcal{U} = \{U_i\}$.

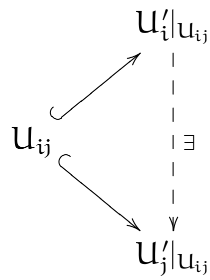
For each i indexing our cover fix a smooth lifting $U_i' \to \mathrm{Spec}\, R[I]$. (There is of course only one up to isomorphism.)

We want to patch these together to get a lifting of X. This data is $X \hookrightarrow X'$, which is the choice of a map

$$\mathcal{O}_{X'} \xrightarrow{I \otimes_R \mathcal{O}_X} \mathcal{O}_X$$

on $|X|$. We have such liftings over each U_i (call them U_i').

On $U_{ij} = U_i \cap U_j$, we get a diagram



We thus get two elements of $\mathrm{Def}_{U_{ij}}(R[I])$. For all i, j, fix an isomorphism $\sigma_{ij} : U'_i|_{U_{ij}} \to U'_j|_{U_{ij}}$.

Note: any other choice of σ_{ij} is given by composing with an automorphsm of

$$U'_i|_{U_{ij}} \leftrightarrow H^0(U_{ij}, T_X \otimes I).$$

There is an obstruction for the σ_i's to glue to an isomorphism $X' \xrightarrow{\sim} X[I]$. Define x_{ij} as the composition

$$x_{ij} : U_{ij}[I] \xrightarrow{\sigma_j^{-1}} U'_{ij} \xrightarrow{\sigma_i} U_{ij}[I]$$

Thus $x_{ij} \in H^0(U_{ij}, T_X \otimes I)$.

Now we need a little lemma.

Lemma. $x_{ik} = x_{ij} + x_{jk}$ in $H^0(U_{ijk}, T_X \otimes I)$.

For this you, you have to just unwind the definition. Here's a sketch of a proof of the lemma.

$$U_{ijk}[I] \xrightarrow{\sigma_k^{-1}} U'_{ijk} \xrightarrow{\sigma_j} U_{ijk}[I] \xrightarrow{\sigma_j^{-1}} U'_{ijk} \xrightarrow{\sigma_i} U_{ijk}[I]$$

commutes, and the map from the first term to the third is x_{jk}, and the map from the third to the fifth is x_{ij}, and finally the map from first to the last is x_{ik}. The commutativity is pretty clear; but you should check that composition corresponds to addition.

Corollary. The $\{x_{ij}\}$ define a Cech cocycle

$$[X'] \in H^{\vee 1}(X, T_X \otimes I) = H^1(X, T_X \otimes I)$$

(At this point we need separatedness to know that Cech cohomology can be computed by *this* open cover.)

Theorem. The map

$$\mathrm{Def}_X(R[I]) \to H^1(X, T_X \otimes I)$$

given by $X' \mapsto [X']$ is an R-module isomorphism.

This is sticky! The fact that the R-module structure is preserved is sticky too.

E-mail address: vakil@math.stanford.edu