DEFORMATION THEORY WORKSHOP: OLSSON 1

ROUGH NOTES BY RAVI VAKIL

Here's the plan

Week 1:

- (1) basic definition
- (2) examples
- (3) obstruction spaces
- (4) examples

Week 2

- (5) Picard categories
- (6) Picard stacks
- (7) truncated cotangent complex (rigorous to here)
- (8) overview of the cotangent complex

Today we'll talk about tangent spaces, except from a functorial point of view.

1. MOTIVATION

Suppose k is an algebraically closed field, and X/k is a scheme of finite type. Fix a closed point $x \in X(k)$. Then the *tangent space of* X *at* x is the dual of the k-vector space m/m^2 where $m \subset \mathcal{O}_{X,x}$ is the maximal ideal. That's the definition you'll find in Hartshorne for example.

But the X that will turn up for us will be a moduli space, and will be given to us as a functor. Then this definition is not so easy to use when you only have the functor.

But we can instead get a hold of the tangent space quite nicely in terms of the functor. To this end, let's introduce dual numbers.

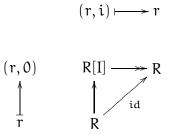
2. DUAL NUMBERS

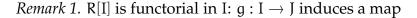
Let R be a ring, and I an R-module. Define R[I], the *ring of dual numbers*, as follows. (Really, there should be some reference to R and I in the terminology, but they will be clear from the context.)

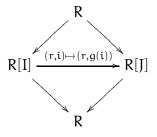
Date: July 23, 2007.

As a group: R/I is $R \oplus I$, and the multiplication rule (the algebra structure) is given by (r, i)(r', i') = (rr', r'i + ri').

I don't just get this ring, I get this diagram:







Remark 2. I = R, write R[ϵ] for R[I]. (This really should be written R[ϵ]/(ϵ ²).

Remark 3. If X is a topological space, O a shaef of rings on X, and I is an O-module, then I can define O[I] in a similar way.

In particular, if X is a scheme, and I is a quasicoherent \mathcal{O}_X -module, then we get a ringed space $X[I] = (|X|, \mathcal{O}_X[I])$.

Exercise. Show that X[I] is a scheme, and we have a closed immersion $X \hookrightarrow X[I]$ and a projection X[I] $\rightarrow X$ composing to give the identity on X:

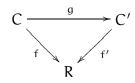


3. Relation with derivations

Suppose $A \to R$ is a ring homomorphism, and M is an R-module. Then an A-*derivation* from R to M is an A-linear map $\partial : R \to M$ such that $\partial(xy) = x\partial(y) + y\partial(x)$. This gives an R-module structure to Der(R, M), the set of all derivations.

So how should you think of it?

Define A-Alg/R as the categorty, where the objects are pairs (C, f) (here C is an A-algebra, and $f : C \to R$ is a map of A-algebras), and the morphisms are $g : C \to C'$ that is compatible with projections to R, i.e. such that



commutes.

Lemma. For any A-derivation ϑ : $R \to I$, the induced map $R \to R[I]$ given by $x \mapsto x + \vartheta(x)$ (by which we really mean $(x, \vartheta(x))$) is a morphism in A-Alg/R and the induced map

$$\operatorname{Der}_{\mathcal{A}}(\mathsf{R},\mathsf{I}) \to \operatorname{Hom}_{\mathcal{A}\text{-}\mathsf{Alg}/\mathsf{R}}(\mathsf{R},\mathsf{R}[\mathsf{I}])$$

is bijective.

We're taking something simple and making it complicated, but we're going to work a lot with this category, so we should get comfortable with how to manipulate it.

We remark that R is viewed as an A-Alg/R by



Let's prove the lemma.

Proof. $R \xrightarrow{s} R[I]$ in A-Alg/R. Then this map must look like $x \mapsto (x, \partial(x))$ for some $\partial(x)$.

This must be a map of A-algebras, hence $\partial(x) = 0$ if x is in the image of A.

It must also be compatible with multiplication. This is the same as saying that given $x, y \in R$, then their product is sent to

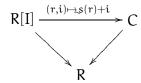
$$(xy, \partial(xy)) = (x, \partial(x))(y, \partial(y)) = (xy, y\partial(x) + x\partial(y)).$$

The necessary compatibility is $\partial(xy) = y\partial(x) + x\partial(y)$ which is exactly the same as saying that δ is a derivation.

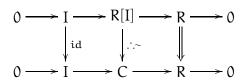
There is a special case: as you know from Hartshorne, you have a universal derivation.

Remark. Suppose we have an object $(f : C \rightarrow R) \in A$ -Alg/R such that I = ker(f) is square-zero, and that f is surjective. Then any section $s : R \rightarrow C$ over A (an A-algebra section), i.e. a morphism from R to C in this category, induces an isomorphism in this

category



This is from



As a special case, $C = R \otimes_A R/J^{\stackrel{f}{2}} \longrightarrow R$ where $J = \ker(R \otimes_A R \rightarrow R, I = J/J^2)$. Define $s : R \rightarrow C$ by $x \mapsto x \otimes 1$. Then s induces an isomorphism

$$\mathbf{R}\otimes_{\mathbf{A}}\mathbf{R}/\mathbf{J}^{2}\cong\mathbf{R}[\Omega_{\mathbf{R}/\mathbf{A}}^{1}].$$

Thus the following map is a canonical bijection

 $\operatorname{Der}_A(R,\Omega^1_{R/A}) \longrightarrow \text{sections of the diagonal map} R \otimes_A R/J^2 \to R$

Question: What is the universal derivation $R \xrightarrow{d} \Omega^1_{R/A}$?

Possible answers: a) $x \mapsto 1 \otimes x - x \otimes 1$, b) $x \mapsto x \otimes 1$. c) $x \mapsto 1 \otimes x$.

Which (if any) is it? It's not obvious! So let's work it out.

According to Hartshorne, $\Omega^1_{R/A} = J/J^2$, and

 $d: \qquad R \longrightarrow \Omega^1_{R/A} = J/J^2$

 $x \longmapsto x \otimes 1 - 1 \otimes x$

Now let's translate this into our language of dual numbers.

$$\begin{array}{c} R[\Omega^{1}_{R/A}] \longrightarrow R \otimes_{A} R/J^{2} \\ s_{d} \\ R \\ R \\ s_{d}(x) = (x, dx) \end{array}$$

Then $1 \otimes x = x \otimes 1 + (1 \otimes x - x \otimes 1)$.

The answer is (c).

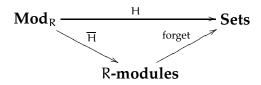
Exercise. Sort this out.

So that's a little bit about dual numbers. Now let's talk about the tangent space of a functor.

4. The tangent space of a functor

 \mathbf{Mod}_{R} the category of finitely generated R-modules. (Interesting question: why restrict to finitely generated R-modules? Answer: that gets imposed in some applications.) H : $\mathbf{Mod}_{R} \rightarrow \mathbf{Sets}$ a functor that commutes with finite products. In other words, $H(I \times J) \rightarrow H(I) \times H(J)$ is an isomorphism for any two modules I and J.

Proposition. Then there is a canonical factorization of my functor H



In other words, with this little extra condition, we get not just sets, but also R-modules. It seems a little miraculous.

Sketch of proof:

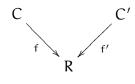
Here's the additive structure. $H(I) \times H(I) \cong H(I \times I)$ induced by $(i, j) \mapsto i + j$. We pull back to H(I).

Multiplicative strucutre: $r \in R$. Then we get a homomorphism of R-modules

$$\cdot f: H(I) \xrightarrow{H(\times f)} H(I)$$

Exercise: Check that this actually gives an R-module structure.

Suppose A \rightarrow is a ring homoorphism. Then A-Alg/R has finite products. Here's why/how:



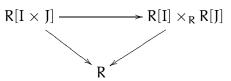
where $(C, f) \times (C', f') = (C \times_R C', (x, y) \mapsto f(x) = f'(y)).$

Lemma. The functor

$\mathbf{Mod}_{R} \rightarrow A\text{-}\mathbf{Alg}/R$

given by $I \mapsto (R[I], \pi : R[I] \rightarrow R)$ commutes with finite products.

Proof. I, $J \in Mod_R$, first take the product in the category of modules and then apply my functor, or else first take my functor and then take product:



The claim is that this should be an isomorphism. This was omitted due to the lack of time. $\hfill \Box$

Thus we get this nice R-module structure.

Corollary. If we are given F : A-**Alg**/ $R \rightarrow$ **Set** such that for $I, J \in Mod_R$, the map $F(R[I] \times_R R[J]) \rightarrow F(R[I]) \times F(R[J])$

is an isomorphism. Then for all $I \in \mathbf{Mod}_{R}$, the set F(R[I]) has a canonical R-module structure.

Reason: F(R[I]) is the image of I under the composition

$$I \longmapsto R[I]$$

$$\mathbf{Mod}_{\mathsf{R}} \longrightarrow \mathsf{A}\operatorname{-Alg}/\mathsf{R} \xrightarrow{\mathsf{F}} \operatorname{Sets}$$

commutes with finite products.

We'll conclude with a definition and then stop.

Definition. Let F : A-Alg/R \rightarrow Sets be a functor satisfying that condition in the Corollary. Then the *tangent space of* F, denoted T_F, is the R-module $F(R[\epsilon])$.

Remark. We don't really need all of A-Alg/R. We only need a full subcategory that contains the image of Mod_R (in the map given in the *Reason* above) and is closed under finite products.

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