DEFORMATION THEORY WORKSHOP: OLSSON 1

ROUGH NOTES BY RAVI VAKIL

Here's the plan

Week 1:

(1) basic definition
(2) examples
(3) obstruction spaces
(4) examples

Week 2

(5) Picard categories
(6) Picard stacks
(7) truncated cotangent complex (rigorous to here)
(8) overview of the cotangent complex

Today we'll talk about tangent spaces, except from a functorial point of view.

1. Motivation

Suppose k is an algebraically closed field, and X/k is a scheme of finite type. Fix a closed point $x \in X(k)$. Then the *tangent space of* X *at* x is the dual of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} \subset \mathcal{O}_{X,x}$ is the maximal ideal. That's the definition you'll find in Hartshorne for example.

But the X that will turn up for us will be a moduli space, and will be given to us as a functor. Then this definition is not so easy to use when you only have the functor.

But we can instead get a hold of the tangent space quite nicely in terms of the functor. To this end, let's introduce dual numbers.

2. Dual numbers

Let R be a ring, and I an R-module. Define $R[I]$, the *ring of dual numbers*, as follows. (Really, there should be some reference to R and I in the terminology, but they will be clear from the context.)

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As a group: R/I is $R \oplus I$, and the multiplication rule (the algebra structure) is given by
$$(r,i)(r',i') = (rr', r'i + ri').$$

I don't just get this ring, I get this diagram:

$$(r,i) \longmapsto r$$

$$\begin{array}{ccc} (r,0) & \quad & R[I] \twoheadrightarrow R \\ \uparrow & & \uparrow \nearrow_{\mathrm{id}} \\ r & & R \end{array}$$

Remark 1. $R[I]$ is functorial in I: $g : I \to J$ induces a map

$$\begin{array}{ccc} & R & \\ \swarrow & & \searrow \\ R[I] & \xrightarrow{(r,i)\mapsto(r,g(i))} & R[J] \\ \searrow & & \swarrow \\ & R & \end{array}$$

Remark 2. $I = R$, write $R[\epsilon]$ for $R[I]$. (This really should be written $R[\epsilon]/(\epsilon^2)$.

Remark 3. If X is a topological space, $\mathcal{O}$ a shaef of rings on X, and I is an $\mathcal{O}$-module, then I can define $\mathcal{O}[I]$ in a similar way.

In particular, if X is a scheme, and I is a quasicoherent $\mathcal{O}_X$-module, then we get a ringed space $X[I] = (|X|, \mathcal{O}_X[I])$.

Exercise. Show that $X[I]$ is a scheme, and we have a closed immersion $X \hookrightarrow X[I]$ and a projection $X[I] \to X$ composing to give the identity on X:

$$\begin{array}{ccc} X & \hookrightarrow & X[I] \\ & \searrow^{=} & \downarrow \\ & & X \end{array}$$

3. Relation with derivations

Suppose $A \to R$ is a ring homomorphism, and M is an R-module. Then an *A-derivation from* R *to* M is an A-linear map $\partial : R \to M$ such that $\partial(xy) = x\partial(y) + y\partial(x)$. This gives an R-module structure to $\mathrm{Der}(R, M)$, the set of all derivations.

So how should you think of it?

Define A-**Alg**$/R$ as the categorty, where the objects are pairs (C, f) (here C is an A-algebra, and $f : C \to R$ is a map of A-algebras), and the morphisms are $g : C \to C'$ that is compatible with projections to R, i.e. such that

$$\begin{array}{ccc} C & \xrightarrow{g} & C' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & R & \end{array}$$

commutes.

Lemma. For any A-derivation $\partial : R \to I$, the induced map $R \to R[I]$ given by $x \mapsto x + \partial(x)$ (by which we really mean $(x, \partial(x))$) is a morphism in A-**Alg**$/R$ and the induced map

$$\mathrm{Der}_A(R, I) \to \mathrm{Hom}_{A\text{-}\mathbf{Alg}/R}(R, R[I])$$

is bijective.

We're taking something simple and making it complicated, but we're going to work a lot with this category, so we should get comfortable with how to manipulate it.

We remark that R is viewed as an A-**Alg**$/R$ by

$$\begin{array}{ccc} R & \xrightarrow{\mathrm{id}} & R \\ \uparrow & \nearrow & \\ A & & \end{array}$$

Let's prove the lemma.

Proof. $R \xrightarrow{s} R[I]$ in A-**Alg**$/R$. Then this map must look like $x \mapsto (x, \partial(x))$ for some $\partial(x)$.

This must be a map of A-algebras, hence $\partial(x) = 0$ if x is in the image of A.

It must also be compatible with multiplication. This is the same as saying that given $x, y \in R$, then their product is sent to

$$(xy, \partial(xy)) = (x, \partial(x))(y, \partial(y)) = (xy, y\partial(x) + x\partial(y)).$$

The necessary compatibility is $\partial(xy) = y\partial(x) + x\partial(y)$ which is exactly the same as saying that δ is a derivation. $\square$

There is a special case: as you know from Hartshorne, you have a universal derivation.

Remark. Suppose we have an object $(f : C \to R) \in A$-**Alg**$/R$ such that $I = \ker(f)$ is square-zero, and that f is surjective. Then any section $s : R \to C$ over A (an A-algebra section), i.e. a morphism from R to C in this category, induces an isomorphism in this category

$$\begin{array}{ccc} R[I] & \xrightarrow{(r,i)\mapsto s(r)+i} & C \\ & \searrow \qquad \swarrow & \\ & R & \end{array}$$

This is from

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \longrightarrow & R[I] & \longrightarrow & R & \longrightarrow & 0 \\ & & \Big\downarrow \text{id} & & \Big\downarrow \therefore\sim & & \Big\Downarrow & & \\ 0 & \longrightarrow & I & \longrightarrow & C & \longrightarrow & R & \longrightarrow & 0 \end{array}$$

As a special case, $C = R \otimes_A R/J^2 \xrightarrow{f} R$ where $J = \ker(R \otimes_A R \to R$. $I = J/J^2$. Define $s : R \to C$ by $x \mapsto x \otimes 1$. Then s induces an isomorphism

$$R \otimes_A R/J^2 \cong R[\Omega^1_{R/A}].$$

Thus the following map is a canonical bijection

$$\mathrm{Der}_A(R, \Omega^1_{R/A}) \xrightarrow{\sim} \text{sections of the diagonal map} R \otimes_A R/J^2 \to R$$

Question: What is the universal derivation $R \xrightarrow{d} \Omega^1_{R/A}$?

Possible answers:
a) $x \mapsto 1 \otimes x - x \otimes 1$,
b) $x \mapsto x \otimes 1$.
c) $x \mapsto 1 \otimes x$.

Which (if any) is it? It's not obvious! So let's work it out.

According to Hartshorne, $\Omega^1_{R/A} = J/J^2$, and

$$d : \qquad \begin{array}{ccc} R & \longrightarrow & \Omega^1_{R/A} = J/J^2 \\ & & \\ x & \longmapsto & x \otimes 1 - 1 \otimes x \end{array}$$

Now let's translate this into our language of dual numbers.

$$\begin{array}{ccc} R[\Omega^1_{R/A}] & \longrightarrow & R \otimes_A R/J^2 \\ s_d \Big\uparrow & & \\ R & & s_d(x) = (x, dx) \end{array}$$

Then $1 \otimes x = x \otimes 1 + (1 \otimes x - x \otimes 1)$.

The answer is (c).

Exercise. Sort this out.

So that's a little bit about dual numbers. Now let's talk about the tangent space of a functor.

4. The tangent space of a functor

$\mathbf{Mod}_R$ the category of finitely generated R-modules. (Interesting question: why restrict to finitely generated R-moodules? Answer: that gets imposed in some applications.) $H : \mathbf{Mod}_R \to \mathbf{Sets}$ a functor that commutes with finite products. In other words, $H(I \times J) \to H(I) \times H(J)$ is an isomorphism for any two modules I and J.

Proposition. Then there is a canonical factorization of my functor H

$$\begin{array}{ccc} \mathbf{Mod}_R & \xrightarrow{\quad H \quad} & \mathbf{Sets} \\ & {\scriptstyle \overline{H}} \searrow \quad \nearrow {\scriptstyle \text{forget}} & \\ & R\text{-}\mathbf{modules} & \end{array}$$

In other words, with this little extra condition, we get not just sets, but also R-modules. It seems a little miraculous.

Sketch of proof:

Here's the additive structure. $H(I) \times H(I) \cong H(I \times I)$ induced by $(i, j) \mapsto i + j$. We pull back to $H(I)$.

Multiplicative strucutre: $r \in R$. Then we get a homomorphism of R-modules

$$\cdot f : H(I) \xrightarrow{H(\times f)} H(I)$$

Exercise: Check that this actually gives an R-module structure.

Suppose $A \to$ is a ring homoorphism. Then A-$\mathbf{Alg}/R$ has finite products. Here's why/how:

$$\begin{array}{ccc} C & & C' \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle f'} & \\ & R & \end{array}$$

where $(C, f) \times (C', f') = (C \times_R C', (x, y) \mapsto f(x) = f'(y))$.

Lemma. The functor

$$\mathbf{Mod}_R \to A\text{-}\mathbf{Alg}/R$$

given by $I \mapsto (R[I], \pi : R[I] \to R)$ commutes with finite products.

Proof. $I, J \in \mathbf{Mod}_R$, first take the product in the category of modules and then apply my functor, or else first take my functor and then take product:

$$\begin{array}{ccc} R[I \times J] & \longrightarrow & R[I] \times_R R[J] \\ & \searrow \qquad \swarrow & \\ & R & \end{array}$$

The claim is that this should be an isomorphism. This was omitted due to the lack of time. □

Thus we get this nice R-module structure.

Corollary. If we are given $F : A\textbf{-Alg}/R \to \mathbf{Set}$ such that for $I, J \in \mathbf{Mod}_R$, the map

$$F(R[I] \times_R R[J]) \to F(R[I]) \times F(R[J])$$

is an isomorphism. Then for all $I \in \mathbf{Mod}_R$, the set $F(R[I])$ has a canonical R-module structure.

Reason: $F(R[I])$ is the image of I under the composition

$$I \longmapsto R[I]$$

$$\mathbf{Mod}_R \longrightarrow A\textbf{-Alg}/R \xrightarrow{F} \mathbf{Sets}$$

commutes with finite products.

We'll conclude with a definition and then stop.

Definition. Let $F : A\textbf{-Alg}/R \to \mathbf{Sets}$ be a functor satisfying that condition in the Corollary. Then the *tangent space of* F, denoted T_F, is the R-module $F(R[\epsilon])$.

Remark. We don't really need all of $A\textbf{-Alg}/R$. We only need a full subcategory that contains the image of $\mathbf{Mod}_R$ (in the map given in the *Reason* above) and is closed under finite products.

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