DEFORMATION THEORY WORKSHOP: LIEBLICH 8

ROUGH NOTES BY RAVI VAKIL

Let's fix a base scheme S locally of finite type over an excellent Dedekind scheme.

For example, S could be Spec of a field.

$\mathcal{F}$ is going to be a stack on S_{ET}, locally of finite presentation. By this we mean that if $A = \lim_{\rightarrow} A_i$, then $\lim_{\rightarrow} \mathcal{F}_{\operatorname{Spec} A_i} \rightarrow \lim \mathcal{F}_{\operatorname{Spec} A}$ is an equivalence of categories. We won't elaborate on this (or define what we mean by the lefthand side).

Brian told us yesterday the following. If $x : \operatorname{Spec} k \rightarrow \mathcal{F}$ admits an effective versal formal deformation (remember what "effective" and "versal" means!!), then there exists a family $X \rightarrow \mathcal{F}$ (finite type over S) such that f is "formally smooth at x".

To be honest, Brian did that with a functor, not with a category. But I'll gloss over that point here.

So there are two separate pieces of content.

1) Schlessinger let us produce a versal formal deformation (indeed a hull). This is purely infinitesimal in nature.

2) Then we have effectivity, which gets us from something formal to something not formal. This was basically Grothendieck's existence theorem. This has more-than-infinitesimal information. This is roughly equivalent to étale-local existence.

(Catchphrase: Effectivity tells us about how algebraic and non-formally-local your moduli problem.)

So if we'd like to make this formal smoothness extend to a neighborhood of X, we want to soup up our deformation theory a bit. Here are some conditions which will enrich Schlessinger's criterion, and the notion of an obstruction theory.

Given $X \rightarrow S$, $a \in \mathcal{F}_X$, let $\mathcal{F}_a$ be the groupod where for each $f : X \rightarrow Y$,

$$(\mathcal{F}_a)_Y = \{\alpha : a \rightarrow b \text{ such that } \operatorname{im}(\alpha) \text{in } S_{ET} \text{ is } f\}.$$

This gets confusing. Max says: "godammit!!"

Artin's global version of Schessinger's criteria are the following.

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(S1′)

$$\begin{array}{ccc} & B & \\ & \downarrow & \searrow \\ A' \twoheadrightarrow & A & A_0 \end{array} \qquad \text{reduced}$$

where A is finite type over S.

$a \in \mathcal{F}_{\operatorname{Spec} A} =: \mathcal{F}(A)$.

Here we want A' A_0 to be an infinitesimal thickening, and that $\ker(A' \to A)$ is an A_0-module.

The condition is that $\mathcal{F}_a(A' \times_A B) \to \mathcal{F}_a(A') \times \mathcal{F}_a(B)$ is an equivalence of categories.

(S2) Suppose M is of finite type. $D_{a_0}(M)$ is a finite A_0-module, with $a_0 = a|_{\operatorname{Spec} A_0}$. Martin: $\overline{\mathcal{F}}(A_0[M]) = D_{a_0}(M)$ a finite (i.e. finitely generated, aka finite type) A_0-module.

These are replacements from the Schlessinger criteria, and in fact specialize to it when we consider an object over a field.

Suppose we are given an obstruction (à la Martin) $A \twoheadrightarrow A_0$ is an infinitesimal extension, $a \in \mathcal{F}(A)$, an obstruction theory

$$O : (A_0\text{-}\mathbf{Mod}_{ft}) \to (A_0\text{-}\mathbf{Mod}_{ft})$$

such that for all $A' \to A \to A0$ deformation situation ($\ker(A' \to A) = M$ is an A_0-module then $o_a(A') \in O_a()M)$ such that $o_a(A') = 0$ iff a lifts to A'.

In addition, we have condition (4.1). (Apologies for Artin's notation!)

(4.1) (i) *Étale localization.* if $A \to B$ is étale, then

$$D_{a_0}(M_0 \otimes B_0) \xleftarrow[\sim]{} D_{a_0}(M_0) \otimes B_0$$

$B_0 = A_0 \otimes_A B$, $M_0 \in A_0\text{-}\mathbf{Mod}_{ft}$,

$$O_{b_0}(M_0 \otimes B_0) \xleftarrow[\sim]{} O_a(M_0) \otimes B_0$$

$b_0 = a_0|_{B_0}$.

(4.1) (ii) *Completion.* If $\mathfrak{m} \subset A_0$ maximal ideal, then

$$D_a(M) \otimes \hat{A}_0 \xrightarrow{\sim} \lim_{\leftarrow} D_{a_0}(M/\mathfrak{m}^n M)$$

(4.1) (iii) *Constructibility.* There exists a dense set of closed points $p \in \operatorname{Spec} A_0$ such that

$$D_{a_0}(M) \otimes k(p) \xrightarrow{\sim} D_{a_0|_{B_0}}(M \otimes k(p))$$

$$O_{a_0}(M) \otimes k(p) \xrightarrow{\sim} O_{a_0|_{B_0}}(M \otimes k(p))$$

Theorem (Artin). Given $\mathcal{F}$, O satisfy (S1), (S2), and (4.1), if $x \in X \xrightarrow{f} \mathcal{F}$, $X \to S$ finite type, f is formally smooth at x, then there exists $U \subset X$, $x \in U$ such that $f|_U : U \to \mathcal{F}$ is formally smooth.

Proposition (Artin). $\mathcal{F}$ is an ARtin stack lcoally of finite type over S if
(1) The diagonal map $\mathcal{F} \to \mathcal{F} \times_S \mathcal{F}$ is representable by algebraic spaces, quasicompact and separated.
(2) (S1′), (S2) hold
(3) If $(\hat{A}, \mathfrak{m})$ is a complete local Noetherian ring over S, then $\mathcal{F}(\hat{A}) \to \lim_{\leftarrow} \mathcal{F}(\hat{A}/\mathfrak{m}^n)$ is an equivalence.
(4) D, O satisfy (4.1).

Example. $\mathcal{M}_g$ is the stack of curves of smooth genus g curves ($g > 1$).

Let's verify the conditions.

(1) $\mathcal{M}_g \to \mathcal{M}_g \times \mathcal{M}_g$. Invoke Grothendieck's proof of the representability of the <u>Isom</u> functor. But we'll soon see a way of checking that.

(2) Schlessinger's criterion is no problem, as Brian essentially showed you (albeit using Schlessinger's original criterion).

(3) Grothendieck's existence theorem applies. How? Because we can stick the curve into projective space using a power of the canonical bundle.

(4) is the interesting one. We want to see that the deformation theory is well-behaved with respect to various sorts of base change.

$$\begin{array}{ccc} \mathcal{C} & & \\ \downarrow & & \\ \operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} A' \end{array}$$

$M = \ker(A' \to A)$.

Recall that $O_{\mathcal{C}}(M) = H^2(\mathcal{C}_{A_0}, T_{C_{A_0}|A_0} \otimes M)$

$D_{a_0}(M) = H^1(\mathcal{C}_{A_0}, T_{\mathcal{C}_A|_{A_0}} \otimes M)$

(i) compatible with etale base change $A_0 \to B_0$ (in Hartshorne, the theormem of formal functions)

(ii) constructibility is cohomology and base change (also in Hartshorne).

$f : \mathcal{C}_0 \to \operatorname{Spec} A_0$. We want

$$R^i f_*(T_{\mathcal{C}_0/A_0} \otimes M) \otimes k(p) \to H^i(\mathcal{C}_p, T_{C_p|p} \otimes M)$$

to be an isomorphism.

Note: no non-trivial infinitesimal automorphisms $(H^0(\mathcal{C}, T))$. this gives you a Deligne-Mumford stack, as we discussed earlier.

In fact there is an even better list than this.

Theorem (Artin). $\mathcal{F}$ is an Artin stack locally of finite type over S if
(1) The Schlessinger-type criteria (S1′), (S2) hold, and ("the tangent space is finite-dimensional") if $a_0 \in \mathcal{F}(A_0)$ and M is a finite A_0-module then $\mathrm{Aut}^{\mathrm{inf}}_{a_0}(A_0[M])$ is a finite A_0-module.
(2) ("Grothendieck existence theorem") $\mathcal{F}(\hat{A}) \to \lim_{\leftarrow} \mathcal{F}(\hat{A}/\mathfrak{m}^n)$ is an equivalence of categories.
(3) D, O, $\mathrm{Aut}^{\mathrm{inf}}_{a_0}(A_0[M])$ satisfy (4.1).
(4) If ϕ is an automorphism of a_0 and $\pi = \mathrm{id}$ at a dense set of points of $\operatorname{Spec} A_0$, then $\phi = \mathrm{id}$.
(5) Now (1)–(4) imply that the diagonal $\mathcal{F} \to \mathcal{F} \times \mathcal{F}$ is representable and separated. Then check that it is quasicompact.

In fact, this if is really an if and only if. But that's hard.

My meta-claim to you is this: if you find yourself in a dark alley with a stack, and you want to show that it is an Artin stack, use this theorem. This is something that people always say is easy in a paper, and it isn't.

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