DEFORMATION THEORY WORKSHOP: LIEBLICH 7

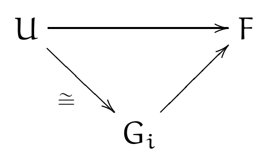
ROUGH NOTES BY RAVI VAKIL

We're ready to talk about algebraic stacks, and take what we've been thinking of into the language of algebraic geometry.

If we're trying to get these topological ideas back in geometry, we are led to the question: What is geometry? One possible answer is that there is some local structure (on top of topology).

Example: Suppose F is a sheaf on S_{fppf} (i.e. a functor with gluing).

Claim. F is a scheme iff there exists a scheme U and a map (of functors) $U \xrightarrow{a} F$ which is Zariski-locally an isomorphism, i.e. there exists a covering $\{G_i \subset F\}$ of open subfunctors, such that for each i, there exists $U_i \subset U$ open with



We can think of a as a *uniformization*.

Definition (temporary). An étale algebraic space over S is a sheaf F on S_{ET} such that there exists a scheme U and a surjective étale represntable morphism $U \to F$. Some hypotheses are necessary. (Perhaps F is locally of finite presentation over S, and $F \to F \times F$ representable, finite type, hence apparently quasi-affine.)

An *fppf algebraic space* is defined in the same way, with "étale" replaced by "fppf".

Theorem (Artin). Any fppf algebraic space is an étale algebraic space. He needs some hypotheses too: the diagonal has to be of finite type.

To be safe, we'll add the hypothesis that the diagonal is of finite type to everything for the rest of this lecture.

Warning: People don't necessarily mean the same thing when they use the phrase "algebraic space".

But you should think of this as: an algebraic space is a sheaf that etale-locally looks like a scheme.

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Example. There exists a smooth three-fold over $\mathbb{C}$ with descent datum with respect to $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ which is not effective. But there does exist a sheaf $\overline{T}/\mathbb{R}$ such that $\overline{T} \otimes \mathbb{C} \cong T$, and $T \to \overline{T}$ is finite etale.

Definition. A stack $\mathcal{X}$ on S_{ET} is a *Deligne-Mumford stack* (or DM-stack) if
(i) $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable (hence all maps from a scheme to X are representable by schemes), quasicompact, and separated
(ii) there exists and etale surjective $X \to \mathcal{X}$ from a scheme.

We can interpret these conditions.

(1) says that for all $f : T \to \mathcal{X}$, $g : T' \to \mathcal{X}$ implies that $\underline{\mathrm{Isom}}(pr_1^* f, pr_2^* g) \to T \times T'$ "is" a quasicompact separated map of schemes.

(2) tells us about the local geometry, but it tells us more. It tells us that the objects must have discrete isomorphism group. Here's why... [skipped]

So in particular, $B\mathbb{G}_m$ is not a Deligne-Mumford stack.

But we still want it to be an algebraic object.

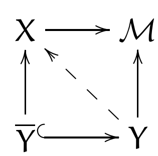
Definition. An *Artin stack* on S_{ET} is a stack $\mathcal{X}$ satisfying:

(i) $\mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable by algebraic spaces, quasicompact. Laumon and Moret-Bailly assume separatedness, but Martin says that it is widely acknowledged that this is not necessary. Max says he'll take it.

(ii) there exists a scheme X and a smooth surjection $X \to \mathcal{X}$.

Theorem (Artin). If $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is representable (by algebraic spaces), quasicompact and separated, then there exists an fppf surjection $X \to \mathcal{X}$ iff there exists a smooth surjection $X' \to \mathcal{X}$, i.e. $\mathcal{X}$ is an Artin stack.

Proposition. Suppose $\mathcal{M}$ is a moduli stack (locally of finite presentation) such that isoms are representable by algebraic spaces, quasicompact and separated. Then $\mathcal{M}$ is an Artin stack iff there exists $X \to \mathcal{M}$ (X locally of finite presentation) which is formally smooth.



Theorem (Artin). An Artin stack $\mathcal{X}$ is Deligne-Mumford iff $\mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is unramified iff no object has non-trivial infinitesimal automorphisms.

There still remains the question of verifying that something is a Deligne-Mumford stack or Artin stack.

Let $S = \operatorname{Spec}\mathbb{C}$.

Example. $\mathcal{M}_{1,1}$, the stack of elliptic curves. Let's define the category. The objects of $(\mathcal{M}_{1,1})_T$ are families $\mathcal{E} \xrightarrow{\pi} T$ with a section σ, such that π is proper and smooth, and for all $\bar{t} \to T$, $g(\mathcal{E}_{\bar{t}}) = 1$.

The condition on Isom's is not so bad. This implies that there are no nontrivial infinitesimal automorphisms.

So by Artin's theorem, it is enough to show that $\mathcal{M}_{1,1}$ is an Artin stack.

The idea for this is to uniformize using the family of plane cubics.

(1) There is a scheme U representing the functor T maps to

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathbb{P}^2_T \\ & \searrow \quad \swarrow & \\ & T & \end{array}$$

smooth families of cubic curves. (One should define what one means by "smooth family of cubic curves".)

We do this as follows. We take the universal cubic.

$$\left(\sum \alpha_{ijk} X^i Y^j Z^k\right) \subset \mathbb{A}^{10} \times \mathbb{P}^2$$

over $\mathbb{A}^{10}$.

There exists $\tilde{U} \subset \mathbb{A}^{10}$ parametrizing smooth cubics. U is the image of $\tilde{U}$ in $\mathbb{P}^9 \longleftarrow \mathbb{A}^{10} \setminus \{0\}$.

(2) There exists a scheme P ($\to U$) representing the functor T mapsto

$$\begin{array}{ccc} \mathcal{C} & \hookrightarrow & \mathbb{P}^2_T \\ & \searrow^{\sigma} & \\ & T & \end{array}$$

along with a section $T \to \mathcal{C}$ [I don't know how to make such diagrams quickly, so I'll say it in words!] of pointed smooth cubics.

(3) We force $\mathcal{O}(1)|_{\mathcal{E}} = \mathcal{O}(3\sigma)|_{\mathcal{E}}$, to get P'

(4) The action of PGL_3 on P' coming from choosing coordinates of $\mathbb{P}^2$.

(5) $[P'/PGL_3] \cong \mathcal{M}_{1,1}$.

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