DEFORMATION THEORY WORKSHOP: LIEBLICH 6

ROUGH NOTES BY RAVI VAKIL

Today we address the question: what makes a stack something that is fundamentally (algebro-)geometric?

This will be an extended meditation on projective space.

Let S be a scheme, $\mathcal{C} = \mathbf{Sch}/S$ (the big étale site).

There are two competing ways of describing projective space $\mathbb{P}^n$. The first is completely rigorous, and the second is only a shadow right now, and our job will be to make it make sense.

(1) $h_{\mathbb{P}^n}(T)\{\ \mathcal{O}^{n+1} \twoheadrightarrow \mathcal{L}\ ,\ \mathcal{L}$ invertible on $T\}/\cong$.

(2) It is the quotient $(\mathbb{A}^{n+1} \setminus \{0\})/\mathbb{G}_m$, where $\mathbb{G}_m$ = multiplicative group, $\mathbb{G}_m(T) = \Gamma(T, \mathcal{O}_T^{\times})$, $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

So let's figure out what (2) means. Let's do our usual thing, of pretending it makes sense, and figure out what would follow from it.

If (2) makes sense, we'd like this to be a principal $\mathbb{G}_m$-bundle, or a $\mathbb{G}_m$-torsor. $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a $\mathbb{G}_m$-torsor. This should be some sort of "unversal $\mathbb{G}_m$-torsor".

So let's map some X to $\mathbb{P}^n$, and pullback the torsor:

$$\begin{array}{ccc} T & \xrightarrow[f]{\mathbb{G}_m\text{-equivariant}} & \mathbb{A}^{n+1} \setminus \{0\} \\ {\scriptstyle \mathbb{G}_m\text{-torsor}}\big\downarrow & & \big\downarrow \\ X & \longrightarrow & \mathbb{P}^n \end{array}$$

Proposition/exercise. There is a natural equivalence of categories { Relative affine X-schemes with $\mathbb{G}_m$-action, and $\mathbb{G}_m$-equivariant maps } $\leftrightarrow$ { $\mathbb{Z}$-graded quasicoherent $\mathcal{O}_X$-algebras with graded maps $\}^{\text{opp}}$.

The idea: Given $Y \xrightarrow{f} X$ gives a $\mathbb{G}_m$-aciton on $f_*\mathcal{O}_Y$ over X. This breaks up as a sum of eigensheaves induced by the characters $\mathbb{Z}$, $\mathbb{G}_m \to \mathbb{G}_m$, $t \mapsto t^m$.

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Example. Action of $\mathbb{G}^m$ on $\mathbb{A}^{n+1}_X = \underline{\mathrm{Spec}}_X \mathcal{O}_X[x_1, \ldots, x_{n+1}]$. We'll take the action $t(a_1, \ldots, a_n) = (t^{-1}a_1, \ldots, t^{-1}a_n)$. Then the action on the algebra is given by $x_i \mapsto tx_i$. The grading is by total degree.

Now $T \to X$ is a $\mathbb{G}_m$-torsor, so this is rleatively affine by descent theory (as $\mathbb{G}_m$ is affine). So this should correspond to some graded sheaf of algebras on X, so let's figure out which graded sheaf of algebras this is.

Proposition. Given a $\mathbb{G}_m$-torsor $T \to X$, there exists and invertible sheaf $\mathcal{L}$ on X such that

$$T \cong \underline{\mathrm{Spec}}_X \oplus_{i\in\mathbb{Z}} \mathcal{L}^{\otimes i}$$

where the action on the left is the natural grading on the right by i.

Proof. fppf-locally on X, $T \cong \underline{\mathrm{Spec}}_X \mathcal{O}_X[x, x^{-1}]$.

The descent datum: graded isomorphism

$$\mathcal{O}[x, x^{-1}] \xrightarrow{\sim} \mathcal{O}[x, x^{-1}]$$

etc.

A $\mathbb{G}_m$-eqvariant map

$$\underline{\mathrm{Spec}} \oplus \mathcal{L}^i = \qquad \begin{array}{ccc} T & \longrightarrow & \mathbb{A}^{n+1} \setminus \{0\} \\ \downarrow & & \downarrow \\ \overline{T} & \longrightarrow & \mathbb{A}^{n+1} \end{array}$$

$$\underline{\mathrm{Spec}} X \oplus_{i\geq 0} \mathcal{L}^i \xrightarrow{\text{graded}} \underline{\mathrm{Spec}}_X \mathcal{O}_X[x_1, \ldots, x_{n+1}]$$

$$\oplus_{i\geq 0}\mathcal{L}^i \twoheadleftarrow_{\text{graded}} \mathcal{O}_X[x_1, \ldots, x_{n+1}]$$

Conclusion. The functor of points tells us is that in fact $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a $\mathbb{G}_m$-torsor.

Let's think harder about this discussion. This even tells us how to take quotients by groups. Suppose G is a group scheme, X a scheme, G acting on X.

We'd love to make a quotient X/G such that $X \to X/G$ is a G-torsor.

Definition. The **quotient stack** $[X/G]$ is defined as follows. I'll tell you what the fiber categories are, and let you figure out what the maps between them are.

The objects are going to be pairs $(T \to Y, \phi)$, where $\phi : T \to x$ is G-equivariant, and the arrows $(T \to Y, \phi) \to (T' \to Y, \phi')$ in the fiber category are commutative diagrams

$$\begin{array}{ccc} T & \xrightarrow{\psi} & T' \\ & & \\ & Y & \end{array}$$

where ψ is a G-equivariant isomorphism.

Note there is a natural map $\nu : X \to [X/G]$:

$$\begin{array}{cc} T & \xrightarrow{\psi} X \\ \downarrow & \\ X & \end{array}$$

We take $T = G \times X$, take the vertical map to be the trivial torsor, and ψ as precisely the G-action.

We expect that the following should hold.

Claim. ν makes X a G-torsor over $[X/G]$.

Proof. This is equivalent to the statement that we have a commutative diagram (where the bottom right corner is a *stack*.

$$\begin{array}{ccc} Y \times_{[X/G]} X & \longrightarrow & X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & [X/G] \end{array}$$

So we have to show that the left side of the square is a G-torsor.

But wait: what is a fiber product of stacks??!

Let's take a time out and make a definition.

Definition. Given morphisms of stacks $\mathcal{X} \xrightarrow{\alpha} \mathcal{Z}$ and $\mathcal{Y} \xrightarrow{\alpha} \mathcal{Z}$. The fiber product has fiber categories

$$(\mathcal{X} \times_{\mathcal{Z}} \mathcal{Y})_T = \{(x, y, \phi) \mid x \in \mathcal{X}_T, y \in \mathcal{Y}_T,\ \phi : \alpha(x) \xrightarrow{\sim} \beta(y) \text{ an arrow in } \mathcal{Z}_T\}$$

Now I should tell you waht the arrows are in the fiber category: $\gamma : x \in x'$, $\delta : y \to y'$, such that

$$\begin{array}{ccc} \alpha(x) & \xrightarrow{\alpha(\gamma)} & \alpha(x') \\ \phi \downarrow & & \downarrow \phi' \\ \beta(y) & \xrightarrow{\beta(\delta)} & \beta(y') \end{array}$$

is commutative.

Example. Let's go back to that diagram we needed to deal with in the proof. Let's make sense of $X \times_{[X/G]} Y$. We need the data of $x \in X(T)$, $y \in Y(T)$.

Things got complicated here, and the diagrams changed too quickly for me to type in.

To make a long story short, $X \times_{\mathcal{Z}} Y \to X \times Y$ may be interpreted as $\underline{\text{Isom}}(pr_1^* f, pr_2^* g)$.

As an important example, consider $X = pt = *$.

$[*/G]_T =$ category of G-torsors. $[*/G] = BG$. $* \to [*/(\mathbb{Z}/2)]$. This is finite etale of degree 2.

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