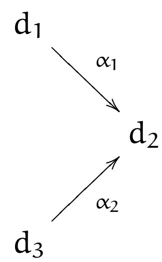
DEFORMATION THEORY WORKSHOP: LIEBLICH 5

ROUGH NOTES BY RAVI VAKIL

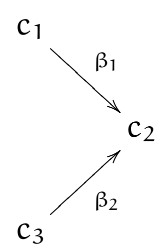
Let's review the definition of a category fibered in groupoids, but using some different language that you might find more appealing.

Definition. A functor $F : \mathcal{D} \to \mathcal{C}$ is a *category fibered in groupoids* (or as some call it, a *groupoid over* C) if

(i) For all $\beta : c_1 \to c_2 \in \mathcal{C}$ and for all $d_2 \in \mathcal{D}$ such that $F(d_2) = c_2$, there exists $\alpha : d_1 \to d_2$ such that $F(\alpha) = \beta$.
(ii) For all



over

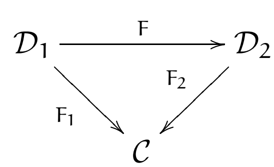


Given $\beta_3 : c_3 \to c_1$ (making the diagram commute), there exists a unique α_3 such that $F(\alpha_3) = \beta_3$.

Definition. Given $c \in \mathcal{C}$, the fiber category $\mathcal{D}_c$ (or F_c) has objects $d \in \mathcal{D}$ such that $F(d) = c$, and the arrows $\alpha : d_1 \to d_2$ such that $F(\alpha) = id_C$.

The last definition I'll write is the following.

Definition. A 1-morphism of cateogires fibered in groupoids $F_1 : \mathcal{D}_1 \to \mathcal{C}$ to $F_2 : \mathcal{D}_2 \to \mathcal{C}$ is a functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ such that diagram below commutes.



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We say F is an *equivalence* if for all $c \in \mathcal{C}$ if for all $c \in \mathcal{C}$, the induced $F_c : (\mathcal{D}_1)_c \to (\mathcal{D}_2)_c$ is an equivalence of categories. (Maybe it is the same as just requiring that F is an equivalence of categories. Martin and Anton both said yes.)

You'll observe that since maps of fibers categories are functors, there are maps between maps. So $\mathrm{Hom}(\mathcal{D}_1, \mathcal{D}_2)$ is a category, and indeed it is a groupoid. Here arrows are natural isomoprhisms between functors.

Now let $\mathcal{C} = \mathbf{Schemes}_S$. We have our old friend $\mathbf{func}(\mathcal{C}^{\mathrm{opp}}, \mathbf{Sets})$, and our other old friend $\mathbf{Schemes}/S$.

Now $\mathbf{Sets} \subset \mathbf{Gpoid}$. Thiis means that our old friends naturally define categories fibered in groupoids.

Example: $\mathcal{D}_1 = h_X$, $X \in \mathbf{Schemes}_S$. Then we have an equivalence of categories

$$\mathrm{Hom}_{\mathcal{C}}(h_X, \mathcal{D}_2) \xrightarrow{\sim} (\mathcal{D}_2)_X$$

Now $\mathcal{M}_{(0)}$ = moduli of varieties.

X scheme, then $\{X \to \mathcal{M}_{(0)}\} \leftrightarrow$

flat families of varieties

$$\begin{array}{c} \mathcal{V} \\ \downarrow \\ X \end{array}$$

Example. $X \mapsto \mathbf{QCoh}(X)$ defines a category fibered in groupoids. (This is the category of quasicoherent sheaves on X, with *isomorphisms* as the arrows.)

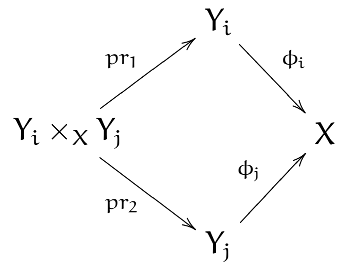
These are supposed to be geometric objects. Do they have some sheaf-like properties? Indeed they do: they satisfy "sheafiness" = gluing = descent theory.

Gluing in general.

Fix $\mathcal{D} \to \mathcal{C} = \mathbf{Sch}_S$. Think of the latter as the (big) étale site.

Definition. Given a covering $\{V_i \to X\}$. The category of descent data (with respect to this covering) is $\mathcal{D}_{\{Y_i \to X\}}$, where the objects are (d, ϕ_{ij}) where $d_i \in \mathcal{D}_{X_i}$, $\phi_{ij} : d_i|_{Y_i \times_X Y_j} \xrightarrow{\sim} d_j|_{Y_i \times_X Y_j}$ $(pr_1^* d_i \to pr_2^* d_j)$ such that $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ for all i, j, k on $Y_i \times_X Y_j \times_X Y_k$. The arrows are what you think they are.

Observation. Any object d of $\mathcal{D}_x$ gives rise to an object of $\mathcal{D}_{\{\psi_i : Y_i \to X\}}$: $d_i = d|_{Y_i} = \psi^*(d)$.



$\psi_i pr_1 = \psi_2 = pr_2$ gives $pr_1^* \psi_i^* \xrightarrow{\sim} pr_2^* \psi_j^*$ which gives $pr_1^* d_i \xrightarrow{\sim} pr_2^* d_j$. Thus the cocycle condition is *built in* to pseudofunctors.

The **upshot** is that we actually get a *functor* $\mathcal{D}_X \to \mathcal{D}_{\{Y_i \to X\}}$.

Now let's say what it means too glue. The additional wrinkle comes because we could glue objects, or we could glue morphisms.

Definition. $\mathcal{D}$ is a *prestack* on $\mathcal{C}$ if $\nu_{\{X_i \to X\}}$ is *fully faithful* for all coverings $\{Y_i \to X\}$ ("descend morhpisms"), and $\mathcal{D}$ is a *stack* if you can glue objects together if $\nu_{\{Y_i \to X\}}$ is an equivalence of cateogires for all $\{Y_i \to X\}$. ("efective descent morhpisms").

The notion of "prestack" will come up in Martin's next lecture. Why remember this notion? Answer: It's kind of like remembering the notion of a "separated presheaf".

Let's reinterpret prestacks. Given $a, b \in \mathcal{D}_X$, define a presheaf $I(a, b)$ on **Schemes**$_X$ as follows. Given $f : Y \to X$, let

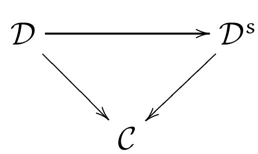
$$I(a, b)(f) := \mathrm{Isom}_{\mathcal{D}_i}(f^* a, f^* b)$$

This defines a presheaf.

"Lemxercise." $\mathcal{D}$ is a prestack iff for all X, a, b, $I(a, b)$ is a sheaf on X_{ET}. "isomoprhisms form a sheaf."

Just as one can sheafify a presheaf, one can stackify a prestack. (In fact, any fibered category.)

Theorem. Given a prestack $\mathcal{D} \to \mathcal{C}$, there exists as tack $\mathcal{D}^s$ and a 1-morphism



such that for all stacks $\mathcal{S} \to \mathcal{C}$, the map $\mathrm{Hom}(\mathcal{D}^s, \mathcal{S}) \to \underline{\mathrm{Hom}}(\mathcal{D}, \mathcal{S})$ is an equivalence of groupoids.

Proposition. QCoh is a stack on $(\mathrm{Spec}\,\mathbb{Z})_{fppf} =$ **Schemes**$/\mathbb{Z}$.

From this you can deduce a host of other things, as you'll see in your exercises.

Proposition. Sheaves on $(\operatorname{Spec}\mathbb{Z})_{\mathsf{ET}}$ form a stack.

This just means that we can glue sheaves together. We've seen this in the Zariski topology, in a Hartshorne exercise.

Let's recall **our moduli problems.** Are they stacks are not?!!

(5) Subspaces of V. They are a stack, because they are a sheaf. (Remark: a sheaf is a stack. A stack is a fancier version of a sheaf. This is an enlightening point, but there is one thing which may confuse you: a sheaf is a stack, and there is a stack of sheaves. The two uses of the words "sheaf" in the previous sentence are completely different uses.)

(4) Closed subschemes of X. This is a stack because it is a sheaf.

(3) How about $\operatorname{Hom}(X, Y)$? Answer: with some work, this is a sheaf, hence a stack.

So why have I given the last few lectures?! Well, because:

(2) Line bundles on X do *not* form a sheaf!!! But it *is* a stack because of descent theory!

(1) Curves of genus not 1: we showed that it is a stack. It is not a sheaf.

(0) Varieties. This is a *prestack* (as $\operatorname{Isom}(X, Y)$ is a sheaf). But it is not a stack!

There are schemes that don't descend.

Exercise: There exists $X/\mathbb{C}$ a smooth threefold (not quasiprojecive), with a descent datum relative to $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$, which does *not* descend.

(Throughout, we are thinking in the BIG ETALE TOPOLOGY for the purposes of concreteness.)

Let me say one last thing here. That last example is kind of funny, as schemes are sheaves. So a family $X \to T$ is a sheaf. But this obsctructed descent data says we can't glue them toeghter as schemes. But we *can* glue them together as sheaves, as sheaves form a stack! So why not just allow us to think of sheaves that are locally like schemes? Why not take the "stacky closure" of **Schemes** in **Sheaves**. Their local structure is just that of schemes. You have just defined the notion of an *algebraic space*! These are the spaces (sheaves) that are etale-locally schemes.

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