DEFORMATION THEORY WORKSHOP: LIEBLICH 4

ROUGH NOTES BY RAVI VAKIL

Let's return to moduli problems. I made a list of problems on the first day, and now might be a good time to come back to that list and reconsider them. Here are the last three.

(3) $\mathrm{Hom}(X, Y)$

(4) Closed subschemes of Y.

(5) subspaces of a fixed vector space V.

We wrote down what we thought the functor of points should be for these problems. Let's check now if these moduli problems $M_{(3)}$ thorugh $M_{(5)}$ are sheaves (in the etale or fppf topologies).

(3). $h_{M_{(3)}}(T) = \mathrm{Hom}_T(X_T, Y_T) = \mathrm{Hom}(X \times T, Y)$.

We prove that Y is a sheaf, so this is a sheaf. Hence $h_{M_{(3)}}$ is an fppf sheaf.

(4). $H_{M_{(4)}}(T) = \{Z \hookrightarrow X \times T, Z \text{ T-flat}\}/\cong$

Well, closed subschemes corresopnd to quasicoherent sheaves $\mathcal{I}_Z \subset \mathcal{O}_{X \times T}$.

Now, isomorphisms are unique if they exist. This means that the sheaf condition is translated into descent data on this inclusion.

fppf descent is effective for quasicoherent sheaf, then these things glue, so $h_{M_{(4)}}$ is a sheaf.

We need both parts of effectivity of descent to make this work.

$$h_{M_{(4)}}(T) \longrightarrow \prod h_{M_{(4)}}(T_i) \rightrightarrows \prod h_{M_{(4)}}(T_i \times_T T_j)$$

The uniqueness of isomorphism means that it is harmless to choose representatives of $\mathcal{I}$. So we have two descent data, as well as a morphism between descent data, so they descend, and the morphism descends.

There's one more thing we need to check: flatness of the glued closed subscheme. We'll say this a little quickly: flatness can be checked fppf-locally (as can other nifty properties, e.g. local freeness). **Lemma.** If $f : X' \to X$ is faithfully flat. A quasicoherent sheaf $\mathcal{F}$ on X is X-flat respecitvely finitely prsented, etc. iff $f^*\mathcal{F}$ is.

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Remark. We could define a new functor, without requiring flatness, and we would *still* get a sheaf. This just turns out to be a bad sheaf...

(5) $h_{M_{(5)}}(T) = \{W \subset \mathcal{O}_T \otimes V \mid \text{ cokernel is locally free}\}$.

Again, isomorphisms are unique if they exist. So the same descent argument applies.

Notice: we are dealing with all of these moduli problems using only a trick or two.

Now let's go back to our first three moduli problems.

(0) Varieties.

(1) Curves of genus g (a special case of (0)).

(2) Line bundles on X.

Now (2) doesn't look that frightening, so let's look at that. Recall that

$$h_{M_{(2)}} = \{\mathcal{L} \text{ on } x \times T\}/\cong$$

which is $\mathrm{Pic}(X \times T)$.

Let's check the sheaf condition for $\{T_i \to T\}$

$$\mathrm{Pic}(X \times T) \longrightarrow \prod \mathrm{Pic}(X \times T_i) \rightrightarrows \prod \mathrm{Pic}(X \times T_i \times_T T_i)$$

Sadly this is exact at neither spot!

Claim. Exactness fails on the left.

Proof. Pic a T such that $\mathrm{Pic}(T) \neq 0$.

Let $\mathcal{M}$ be a non-trivial sheaf on T.

Then $p_2^*\mathcal{M} \in \mathrm{Pic}(X \times T)$, where p_2 is the projection $X \times T \to T$.

We can choose a *Zariski*-open $\{T_i \subset T\}$ such that $\mathcal{M}$ is trivial on this cover. Then so long as $p_2^*\mathcal{M}$ isn't trivial on $X \times T$ (make an example where this works! e.g. X a point...) then we have two line bundles that are locally the same that are not the same. □

Exactness in the middle is wrong for a more interesting reason.

Claim. Exactness fails at the middle (in general).

Proof. Let $X/\mathbb{R} : (x^2 + y^2 + z^2 = 0) \subset \mathbb{P}^2_{\mathbb{R}}$

We know: $X \otimes_{\mathbb{R}} \cong \mathbb{P}^1_{\mathbb{C}}$, but X is not congruent to $\mathbb{P}^1_{\mathbb{R}}$. Thus there are no divisors of degree 1. (Exercise: Use Riemann-Roch to show this if you haven't seen this fun fact before!)

Consider the covering $\mathrm{Spec}\,\mathbb{C} \to \mathrm{Spec}\,\mathbb{R}$.

Then the exact sequence we are considering is:

$$\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X \otimes \mathbb{C}) \rightrightarrows \mathrm{Pic}(X \otimes \mathbb{C} \otimes \mathbb{C})$$

which (given that $\mathrm{Spec}\,\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathrm{Spec}\,\mathbb{C} \coprod \mathrm{Spec}\,\mathbb{C}$) is

$$\mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \rightrightarrows \mathbb{Z} \times \mathbb{Z}$$

Somehow the problems come from the fact that isomorphisms are not unique. And in (3)–(5) above, we had no problems precisely because isomorphisms are unique.

If you try to make this precise, you end up concluding that descent fails because there we have lcoal line bundles $\mathcal{L}$ on $X \times T'$, then $p_1^*\mathcal{L} \xrightarrow{\sim} p_2^*\mathcal{L}$ on $X \times T''$, yet (in Martin's language) $\phi_{jk} \circ \phi_{ij} \neq \phi_{ik}$.

To fix the problem, we think about categories instead of sets.

Definition. A *groupoid* is a category where every arrow is an isomorphism.

So given a category, you can get a groupoid by keeping all your objects, and only those morphisms that are isomorphisms.

Definition. A groupoid $\mathcal{C}$ is *discrete* if for each $x \in \mathbb{C}$, $\mathrm{Aut}(x)$ is the identity. It is *connected* if any two objects are isomorphic.

We have a functor χ : **Set** $\to$ **Groupoid**.

Lemma. The essential image of χ is the discrete groupoids.

Here are more good things:

$M_{(2))}(T)$ is a groupoid. {groupoid of $\mathcal{L}$ on $X \times T$ }.

$S \xrightarrow{f} T$.

Then $M_{(2)}(T) \longrightarrow M_{(2)}(S)$ a functor. $\mathcal{L}$ on $X \times T$ maps to $(1 \times f)^*\mathcal{L}$ on $X \times S$.

We mght guess that we have a contrvariant functor $M_{(2)}$: **Sch**^{o} $\to$ **Groupoids**.

Now $T'' \xrightarrow{g} T' \xrightarrow{f} T$. There exists an isomorphism $g^*f^* \xrightarrow{\sim} (fg)^*$. this is from the unversal property of pullback — pullbacks are unique up to unique isomorphism.

Exercise: what does the pullback mean?

Now given $T''' \xrightarrow{h} T'' \xrightarrow{g} T' \xrightarrow{f} T$. Then we have a commutaive diagram of isomorphisms of functors.

(1)
$$\begin{array}{ccccc} & & h^*(fg)^* & & \\ & \overset{\cong}{\nearrow} & & \overset{\cong}{\searrow} & \\ h^*g^*f^* & & & & (fgh)^* \\ & \underset{\cong}{\searrow} & & \underset{\cong}{\nearrow} & \\ & & (gh)^*f^* & & \end{array}$$

This collection of information has a name.

Definition. A *fibered category with clivage*, (or a *pseudo-functor*) over a category $\mathcal{C}$, is:

(1) for each $c \in \mathbb{C}$, a groupoid $F(c)$

(2) for each arrow $f : c \to d$ in $\mathcal{C}$, a functor $f^* : F(d) \to F(c)$, and

(3) for each pair of arrows $c \xrightarrow{f} d \xrightarrow{g} e$ an isomorphism $\nu_{f,g} : f^*g^* \to (gf)^*$ such that (1) commutes.

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