DEFORMATION THEORY WORKSHOP: LIEBLICH 3

ROUGH NOTES BY RAVI VAKIL

Today we'll talk about descent theory, which is a fancy word for gluing.

As motivation, consider what gluing means in "Zariski-land": Suppose X is a shceme, and $\{U_i \subset X\}$ is an open covering, and $\mathcal{F}_i$ on U_i is quasicoherent sheaf on U_i. Suppose we're given an isomorphism

$$\phi_{ij} : \mathcal{F}_i|_{U_i \cap U_j} \xrightarrow{\sim} \mathcal{F}_j|_{U_i \cap U_j}$$

where $\phi_{jk} \circ \phi_{ij} = \phi_{ik}$ for all i, j, k on $U_i \cap U_j \cap U_k$.

Defintion. Suppose $f : X' \to X$ is an fpqc morphism.

$$X'' := \quad \begin{array}{ccc} & X' \times_X X' & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X' & & X' \end{array}$$

and $\mathcal{F}'$ is a quasicoherent sheaf on X'.

A *descent datum* is an isomorphism

$$\phi : p_1^* \mathcal{F}' \xrightarrow{\sim} p_2^* \mathcal{F}'$$

such that $p_{23}^* \phi \circ p_{12}^* \phi = p_{13}^* \phi$, (here p_{ij} is a projection $X' \times_X X' \times_X X' \to X' \times_X X'$ "onto the ith and jth components") i.e. that the following frightening diagram commutes

$$\begin{array}{ccccc} p_{12}^* p_1^* \mathcal{F}' & \xrightarrow{p_{12}^* \phi} & p_{12}^* p_2^* \mathcal{F}' & & \\ \| & & \| & & \\ \| & & p_{23}^* p_1^* \mathcal{F}' & \xrightarrow{p_{23}^* \phi} & p_{23}^* p_2^* \mathcal{F}' \\ \| & & & & \| \\ p_{13}^* p_1^* \mathcal{F} & & \xrightarrow{\quad p_{23}^* \phi \quad} & & p_{13}^* p_2^* \mathcal{F}' \end{array}$$

Re-interpretation in terms of the "functor of points":

Definition'. A descent datum on $\mathcal{F}'$ consists of an isomorphsm

$$\phi_{t_1, t_2} : t_1^* \mathcal{F}' \xrightarrow{\sim} t_2^* \mathcal{F}'$$

for all $t_1, t_2 \in X'(T)$, fixed $T \in \mathbf{Sch}_X$ (i.e. a diagram $T \to X$)

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such that for all $t_1, t_2, t_3 \in X'(T)$,

$$\phi_{t_2,t_3} \circ \phi_{t_1,t_2} = \phi_{t_1,t_3}$$

(which implies that $\phi_{t,t} = \mathrm{id}$), **and** this is functorial in T, t_i.

Note. If $\mathcal{F}' = f^*\mathcal{F}$, there is a natural descent datum

$$\phi_{t_1,t_2} : t_1^* f^* \mathcal{F} \xrightarrow{\sim} t_2^* f^* \mathcal{F}$$

where $ft_1 = ft_2$ (because T is an X-scheme), so

$$\begin{array}{ccc} (ft_1)^* & = & (ft_2)^* \\ \downarrow \cong & & \downarrow \cong \\ t_2^* f^* & \xrightarrow{\sim} & t_2^* f^* \end{array}$$

Definition. The *category of descent data for* f, $\mathcal{D}_f$, the category of pairs $(\mathcal{F}', \phi)$ where $\mathcal{F}'$ is a quasicoherent sheaf on X' is a descent datum. Call this $(f^*\mathcal{F}, \mathrm{can})$ (here "can" is for "canonical").

Maps: $\phi : \mathcal{F}_1' \to \mathcal{F}_2'$ such that

$$\begin{array}{ccc} p_1^* \mathcal{F}_1' & \xrightarrow{\phi_1} & p_2^* \mathcal{F}_1' \\ \downarrow & & \downarrow \\ p_1^* \mathcal{F}_2' & \xrightarrow{\phi_2} & p_2^* \mathcal{F}_2' \end{array}$$

Note: pullback defines a functor $\tilde{f^*} : \mathbf{QCoh}(X) \to \mathcal{D}_f$ given by $\mathcal{F} \mapsto (f^*\mathcal{F}, \mathrm{can})$.

Definition. f is a *descent morphism* if $\tilde{f^*}$ is fully faithful. f is an *effective descent morphism* if $\tilde{f^*}$ is an equivalence.

In plain english (which is admittedly less precise than the categorical language): Given quasicoherent sheaf on X, we'll get a quasicoherent sheaf on X' that will automatically satisfy some properties (i.e. that a certain diagram commutes), i.e. it will give us a descent datum. This behaves well with respect to moprphisms in $\mathbf{Qcoh}(X)$ (this is the statement that we get a functor from $\mathbf{Qcoh}(X)$ to the category of descent data $\mathcal{D}_f$). f is a *descnet morphism* if every map between descent datum coming from quasicoherent sheaves comes from an honest map between sheaves. f is an *effective descent morphism* if every descent datum (something that looks like it comes from a shef downstairs) actually *does* come from a quasicoherent sheaf on X (or more precisely, is isomoprhismc to something coming from a quasicoherent sheaf on X).

David asked: if f is a proper birational map of varieties, is f a descent morphism? The answer is yes, according to some result of Raynaud.

Theorem (Grothendieck). If $f : X' \to X$ is fpqc (faithfully flat and quasicompact), then f is an effective descent morphism for quasicoherent sheaves.

In other words: *we can glue!*

Theorem (Giraud/Grothendieck). If f has a section, then f is an effectiev descent morphism. We have no requriemetns on f other than this!

Proof. Suppose σ is our section.

Now $\tilde{f}^*$ is an equivalence means: $\tilde{f}^*$ $\tilde{f}^*$ is *essentially surjective*. (This means that every object of $\mathcal{D}_f$ is *isomorphic to* $\tilde{f}^*$ of something.) So let's check this.

$\tilde{f}^*$ is clearly faithful $\sigma^* f^* = \mathrm{id}$. $\alpha : \mathcal{F} \to \mathcal{G}$ such that $f^*\alpha = 0$ implies $\sigma^* f^* \alpha = \alpha = 0$.

It is full: a map $(\mathcal{F}, \phi) \xrightarrow{\psi} (\mathcal{F}', \phi')$ is the same as the information

$$\begin{array}{ccc} & & X' \\ & \overset{t}{\nearrow} & \downarrow \\ T & & \\ & \searrow & \downarrow \\ & & X \end{array}$$

such that $t^*\mathcal{F} \xrightarrow{\phi_t} t^*\mathcal{F}'$ such that for all t_1, t_2,

$$\begin{array}{ccc} t_1^*\mathcal{F} & \xrightarrow{\phi_{21}} & t_2^*\mathcal{F}' \\ {\scriptstyle \phi_{t_1,t_2}}\downarrow & & \downarrow{\scriptstyle \phi'_{t_1,t_2}} \\ t_2^*\mathcal{F} & \xrightarrow{\phi_{21}} & t_2^*\mathcal{F}' \end{array}$$

commutes.

Now the reference point σ_T (i.e. $T \longrightarrow X \xrightarrow{\sigma} X'$) we have

$$\begin{array}{ccc} \sigma^*\mathcal{F} & \xrightarrow{\psi_\sigma == \sigma^*\psi} & \sigma^*\mathcal{F}' \\ {\scriptstyle \sigma\phi}\downarrow & & \downarrow{\scriptstyle \phi'_{\sigma,t}} \\ t^*\mathcal{F} & \xrightarrow{\psi_t} & t^*\mathcal{F}' \end{array}$$

which means we can we can propagate ϕ_σ.

We now check essential surjectivity. $(\mathcal{F}, \phi) \in \mathcal{D}_f$. $t \in X'(T)$, $T \in \mathbf{Sch}_X$. We hope

$$(\mathcal{F}_f) \cong \widetilde{f^*}(\sigma^*\mathcal{F})$$

$\phi_{t,\sigma f t} : t^*\mathcal{F} \to t^* f^* \mathcal{F}$

$$\begin{array}{ccc} (\mathcal{F}, \phi) & \cong & \widetilde{f^*}(\sigma^*\mathcal{F}) \\ & \searrow & \| \\ & & (f^*\sigma^*\mathcal{F}, \mathrm{can}) \end{array}$$

Given t_1, t_2, the following diagram commtues.

$$\begin{array}{ccc} t_1^*\mathcal{F} & \longrightarrow & t_1^* f^* \sigma^* \mathcal{F} \\ \downarrow{\scriptstyle \phi_{t_1,t_2}} & & \downarrow \\ t_2^*\mathcal{F} & \longrightarrow & t_2^* f^* \sigma^* \mathcal{F} \end{array}$$

and this commutativity comes from gluing for $\mathcal{F}$. (This requires some private thought.) We're using $\sigma f t_1 = \sigma f t_2$. This ends the proof of the theorem of Giraud and Grothendieck, of descent for morphisms with section.

Let's now prove a special case of Grothendieck's theorem, in the case where X and X' are affine (say $\operatorname{Spec} B \to \operatorname{Spec} A$, so $A \to B$ is faithfully flat).

Let's check (a) that $\tilde{f^*}$ is fully faithful. In other words, we want to check that if M and N are A-modules, we want to show that

$$\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_B(M \otimes_A B, N \otimes_A B) \rightrightarrows \operatorname{Hom}(M \otimes_A B \otimes_A B, N \otimes_A B \otimes_A B)$$

is exact.

Recall (adjointness) that we have natural isomorphism $\operatorname{Hom}_B(P \otimes_A B, Q) \to \operatorname{Hom}_A(P, Q)$. (Please ask if you've not seen this before!) Then the desired exact sequence is precisely

$$\operatorname{Hom}_A(M, N) \longrightarrow \operatorname{Hom}_A(M, N \otimes_A B) \rightrightarrows \operatorname{Hom}_A(M, N \otimes_A B \otimes_A B)$$

which is:

$$\operatorname{Hom}_A(M, \; N \longrightarrow N \otimes_A B \rightrightarrows N \otimes_A B \otimes_A B$$

It sufficeds to check that

$$N \longrightarrow N \otimes_A B \rightrightarrows N \otimes_A B \otimes_A B$$

is exact. Here's why: Hom is left exact, and checking the exactness of the above equation display can be restated as checking the (left-)exactness of

$$0 \longrightarrow N \longrightarrow N \otimes_A B \longrightarrow N \otimes_A B \otimes_A B$$

where the right arrow is the difference of the two corresponding arrows in the previous diagram).

To show this, as in yesterday's homework, reduce to the case where there is an *augmentation* $B \to A$ and follow your nose.

Let's now check that $\tilde{f^*}$ is essentially surjective. Given $(\mathcal{F}, \phi) \to M$. $\phi : B \otimes_A M \xrightarrow{\sim} M \otimes_A B$ as $B \otimes_A B$-modules. We guess what $\mathcal{G}$ on X should be, such that $\tilde{f^*}(\mathcal{G}) \cong (\mathcal{F}, \phi)$. $N = \{m \in M \mid m \otimes 1 = \phi(1 \otimes m)\}$.

Observation: there exists a map $\nu : N \otimes_A B \to M$ which is "compatible with descent".

Goal: show this is an isomoprhism. Grothendieckian trick: it suffices to do this after a faithfully flat base change. So we may assume there exists an augmetnation $B \to A$, i.e. a section $X \to X'$ of $f : X' \to X$.

Now we KNOW that descent is effective. So the profo in this case shows that ν is an isomorphism in $\mathcal{D}_f$.

So we have completed the "affine case" of Grothendieck's theorem.

We don't have time to discuss the general case.

About subtle foundational issues (small categories, set-theoretic issues, etc.): It is true that some people should be careful about these isues: but do you really want to be one of those people?

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