DEFORMATION THEORY WORKSHOP: LIEBLICH 2

ROUGH NOTES BY RAVI VAKIL

Suppose X is a scheme. Then the associated contravariant functor h_X is a sheaf, in the following sense. By this we mean the following. Fix Y. Then if we associate $\mathrm{Hom}(U, X) = h_X(U)$ for each o pen set $U \subset Y$, then this defines a sheaf in the Zariski topology. This says nothing more than the fact that morphisms to X glue.

We can say this in a more convenient matter. If $\{U_i \subset Y\}$ is an open cover, then

$$h_X(Y) \xrightarrow{a} \prod_j h_X(U_i) \overset{b}{\underset{c}{\rightrightarrows}} \prod_{i,j} h_X(U_i \cap U_j)$$

is exact, by which we mean that a is injuective, and $\mathrm{im}(a) = \{\alpha \mid b(\alpha) = c(\alpha)\}$.

Now there is a 'problem with the Zariski topology: it is not "geometric". What do we mean by this? In Serre's FAC, for example, we learned that that coherent sheaves in the Zariski topology behave like the coherent sheaves in the classical topology. But for example, the Zariski topology can't see: the fundamental group.

Grothendieck had the insight that we can generalize the notion of topology. Notice: if X is a topological space, then we can understand the topology in terms of the "category of open sets". The objects are the open sets $U \subset X$. We have one arrow $U \to V$ for each inclusion $U \subset X$. (In particular, $|\mathrm{Hom}(U, V)| \leq 1$ — there is at most one morphism from U to V.)

Then already we can say what a presheaf is: it is a contravariant functor to **Set**.

For sheaves, we need more information: we need to remember what covering is. So we retain the information $\{V_i \subset U\}$, some set of arrows, that we declare to be a covering.

There are three silly properties of coverings that we need:

(i) Each open set covers itself: $\{U \subset U\}$ is a cover.
(ii) Coverings pull back: If $\{V_i \subset U\}$ is a covering, and $W \subset U$, then $\{V_i \cap W \subset W\}$ is a cover.
(iii) Coverings compose: If $\{W_{ij} \subset V_i\}$ are coverings, and $\{V_i \subset U\}$ is a covering, then $\{W_{ij} \subset U\}$ is a covering.

Based on this, we generalize the notion of topology so that sheaves still make sense.

Definition. Given a category $\mathcal{C}$, a *Grothendieck topology* iis a collection of sets of arrows $\{V_i \to U\}$ for each $U \subset \mathcal{C}$ (called "coverings") such that:

Date: July 24, 2007.

(i) Any isomorphism is a coverinng.
(ii) Coverings pull back: If $\{V_i \to U\}$ is a covering and $W \to U$, then $V_i \times_U W$ exists for each i, and $\{V_i \times_U W \to W\}$ is a covering.
(iii) Coverings compose: If $\{W_{ij} \to V_i\}$ are coverings, and $\{V_i \to U\}$ is a covering, then $\{W_{ij} \to U\}$ is a covering.

A *site* is a category wiht a Grothendieck topology.

Example. Suppose X is a scheme. X_{Zar}, the *small Zariski topology*, is defined as follows. The objects are open immersions into X, and the arrows are maps that commute with the immersion maps. The coverings are collections of arrows $V_i \to X$ such that the images of the V_i cover the target. This is precisely our earlier definition.

Example. X_{ZAR}, the *big Zariski topology*, is defined as follows. The objects are X-schemes, and the arrows are maps *that are open immersions* that cover the target.

One example of the a difference between the big Zariski site and the small Zariski site is the following. Consider the closed immersion of a point Y into $\mathbb{A}^1 = X$. Then h_Y is a presheaf on both the big and small Zariski-sites. It is representable in X_{ZAR}, but not in X_{Zar}.

Example. $X_{\text{ét}}$, the small étale site, is defined as follows.

$$\mathcal{C} = \{Z \to X \text{ étale}\} \subset \mathbf{Sch}_X.$$

The coverings are

$$\left\{ \begin{array}{ccc} Y_i & \xrightarrow{\phi_i} & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array} \;\middle|\; \cup \phi_i(Y_i) = Z \right\}$$

Note: Each ϕ_i is forced to be étale.

Example. The big étale site is defined analogously.

The mother of sites is the following.

Example. X_{fppf}, the "fppf" site, is the following. ("fppf" means "faithfully flat and locally of finite presentation".)

$$\mathcal{C} = \mathbf{Sch}_X,$$

and coverings are

$$\left\{ \begin{array}{ccc} Y_i & \xrightarrow{\phi_i} & Z \\ & \searrow \quad \swarrow & \\ & X & \end{array} \;\middle|\; \cup \phi_i(Y_i) = Z \text{and } \phi_i \text{ is flat and locally of finite presentation} \right\}$$

Interesting question: This is the "big fppf site". Why don't we consider the "small fppf site"? One problem with that site: There aren't arbitrary fiber products in that category.

Definition. Given a site $\mathcal{C}$, a *sheaf* (of sets) on $\mathcal{C}$ is a functor $F : \mathcal{C}^{\text{opp}} \to \mathbf{Set}$ such that for all coverings $\{Y_i \subset Z\}$ in $\mathcal{C}$, the diagram

$$h_X(Y) \xrightarrow{a} \prod_j h_X(U_i) \underset{c}{\overset{b}{\rightrightarrows}} \prod_{i,j} h_X(U_i \cap U_j)$$

is exact.

Warning: unlike the Zariski topology, we do *not* require $i \neq j$ in the above diagram.

Given a functor, you could ask: is it a sheaf in the Zariski topology? Is it a sheaf in the étale topology? Is it a sheaf in the fppf topology? These are increasingly strict conditions. We have already seen that h_X is a sheaf in the Zariski topology. In fact:

Theorem (Grothendieck). For any X-scheme Y, the functor $h_Y : \mathbf{Sch}_X^{\text{opp}} \to \mathbf{Set}$ is an fppf-sheaf.

You'll see that we never use the "locally of finite presentation" part of the definition, only the "faithfully flat" part.

Let's prove this now.

Let's first understand the simplest possible case. Suppose the cover $\{Y_i \to Z\}$ is $\operatorname{Spec} B \to \operatorname{Spec} A$, i.e. $A \to B$ is a faithfully flat ring extension. Suppose also $Y = \operatorname{Spec} C$. The diagram becomes

$$\operatorname{Hom}(C, A) \xrightarrow{a} \operatorname{Hom}(C, B) \underset{c}{\overset{b}{\rightrightarrows}} \operatorname{Hom}(C, B \otimes_A B)$$

This can be interpreted as

$$\operatorname{Hom}(C,\ A \longrightarrow B \underset{c}{\overset{b}{\rightrightarrows}} B \otimes_A B\ ).$$

Lemma.

$$\begin{array}{ccc} b & \mapsto & b \otimes 1 \\ A \longrightarrow B & \rightrightarrows & B \otimes_A B \\ b & \mapsto & 1 \otimes b \end{array}$$

is exact.

Thiis is equivalent to: $0 \to B \to B \otimes_A B$ is an exact sequence of A-modules, where that latter map is $b \mapsto b \otimes 1 - 1 \otimes b$.

We first deal with the special case where there exsits a map $\sigma : B \to A$ such that the composition $A \to B \to^{\sigma} A$ is the identity.

Then we get $B \otimes_A B \to B$ given by $b \otimes c \mapsto \sigma(b)c$. Then show: if $b \otimes 1 = 1 \otimes b$, then $b \in A$. That implies $\sigma(b) = b$, and if the first is in A then so is the second, and we're done. So if we could only be in that case, we'd be very happy!

But observe: to prove that $0 \to A \to B \to B \otimes_A$ Bis exact, it suffices to prove it after a faithflly flat base change $A \to D$. There's only one faithfully flat map in our picture: $A \to B$. So we'll take $D = B$. Then $A \to B$ turns into $B \to B \otimes_A B$, given by $b \mapsto b \otimes 1$. This has an augmentation: multiplication!

(To make this complete, you have to check that if you take the base change of the diagram, you get the diagram of the base change.)

So we're done this lemma!

Lemma. $F : \mathbf{Sch}_X^{opp} \to \mathbf{Set}$ is an fppf sheaf iff (1) F is a Zariski sheaf.

(2) For all $\operatorname{Spec} B \to \operatorname{Spec} B$, $A \to B$ fppf, the sequence is exact.

We'll omit this. But we have an immediate

Corollary. If Y is affine, then h_Y is an fppf sheaf.

Sketch of general case (Y arbitrary). Let $Y_i \subset Y$ be an affine covering. Suppose $U \to V$ is an fppf covering. We want to check that

$$h_S(V) \longrightarrow h_S(U) \rightrightarrows h_S(V)$$

is exact. fppf-ness comes up here.

Max sketched what to do here.

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