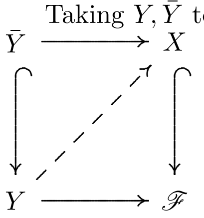
1 Lieblich

Let S be a scheme locally of finite type over an excellent Dedekind scheme.

$\mathscr{F}$ is a stack on S_{ET}, and we will assume that it is locally of finite presentation. That is, $A = \varinjlim A_i$ is a ring, then $\varinjlim \mathscr{F}_{\operatorname{Spec} A_i} \to \mathscr{F}_{\operatorname{Spec} A}$ is an equivalence of categories.

Brian said that if x admits an effective versal formal deformation then there exists $\operatorname{Spec} k \to X \xrightarrow{f} \mathscr{F}$ such that f is formally smooth at x (X is of finite type over S)

Taking $Y, \bar{Y}$ to be local Artin schemes, we get



pt of $\bar{Y}$ maps to x.

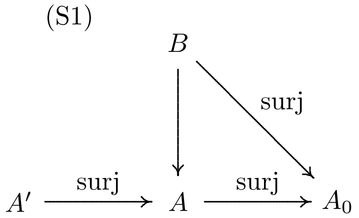
The content is the following:

1. Schlessinger implies that there exists a versal formal deformation (hull)
2. formal to effective via GET, using étale local existence.

Given $X \to S$, $a \in \mathscr{F}_X$ let $\mathscr{F}_a$ be the stack on Sch_X, $f : X \to Y$, $(\mathscr{F}_a)_Y = \{\alpha : a \to b$ such that $\operatorname{Im}(\alpha)$ in S_{ET} is $f\} = \{b \in \mathscr{F}_Y, \varphi : a \to f^*b\}$.

So $\bar{\mathscr{F}}_a(Y)$ =isomorphism classes of $(\mathscr{F}_a)_Y$.

(S1)



Where the bottom row are small thickenings and $\ker(A' \to A)$ is an A_0-module.

Take $a \in \mathscr{F}_{\operatorname{Spec} A} = \mathscr{F}(A)$.

$\mathscr{F}_a(A' \times_A B) \to \mathscr{F}_a(A') \times \mathscr{F}_a(B)$ is an equivalence of categories.

(S2): M is of finite type as an A_0-module, $D_{a_0}(M)$ is a finite A_0-module, $a_0 = a|_{\operatorname{Spec} A_0}$.

Martin said that $\bar{\mathscr{F}}_a(A_0[M]) = D_{a_0}(M)$ an A_0-module

Suppose we're given an obstruction (a la Martin) $A \to A_0$ a small thickening, $a \in \mathscr{F}(A)$, $\mathscr{O}_a : (A_0 - Mod_{ft}) \to (A_0 - mod_{ft})$ such that $A' \to A \to A_0$ is a deformation situation, $\ker(A' \to A) = M$ is an A_0-module, then $o_a(A') \in \mathscr{O}_a(M)$ such that $o_a(A') = 0$ iff a lifts to A'.

In addition, we assume (with $A \to A_0$ a small thickening) conditions called (4.1) (all tensor products over A_0)

1. Etale localization: if $A \to B$ is étale, then $D_{a_0}(M_0 \otimes B_0) \leftarrow D_{a_0}(M_0) \otimes B_0$ isomorphism, $B_0 = A_0 \otimes_A B$, $M_0 \in A_0 - mod_{ft}$, $\mathscr{O}_{b_0}(M_0 \otimes B_0) \leftarrow \mathscr{O}_{a_0}(M_0) \otimes B_0$ is an isomorphism with $b_0 = a_0|_{B_0}$.
2. Completion: If $\mathfrak{m} \subset A_0$ is maximal, then $D_{a_0}(M) \otimes \hat{A}_0 \simeq \varprojlim D_{a_0}(M/\mathfrak{m}^n M)$
3. Constructibility: There exists a dense set of closed points $p \in \operatorname{Spec} A_0$ such that $D_{a_0}(M) \otimes k(p) \simeq D_{(a_0)_p}(M \otimes k(p))$ and $\mathscr{O}_{a_0}(M) \otimes k(p) \simeq \mathscr{O}_{(a_0)_p}(M \otimes k(p))$.

Theorem 1 (Artin). *Given $\mathscr{F}, \mathscr{O}$ satisfying S1,S2 and 4.1, if $x \in X \xrightarrow{f} \mathscr{F}$, $X \to S$ of finite type, f formally smooth at x, then there exists $U \subset X$, $x \in U$ such that $f|_U : U \to \mathscr{F}$ is formally smooth.*

Proposition 1 (Artin). *$\mathscr{F}$ is an Artin stack locally of finite type over S if*

1. $\mathscr{F} \to \mathscr{F} \times \mathscr{F}$ is representable by algebraic spaces, quasi compact, separated.

2. S1 and S2 hold.

3. If $(\hat{A}, \mathfrak{m})$ is a complete local noetherian ring over S, then $\mathscr{F}(\hat{A}) \to \varprojlim \mathscr{F}(\hat{A}/\mathfrak{m}^n)$ is an equivalence.

4. $D, \mathscr{O}$ satisfy 4.1

Example 1. *$\mathscr{M}_g$ the stack of curves of genus g for $g > 1$.*

1. $\mathscr{M}_g \to \mathscr{M}_g \times \mathscr{M}_g$ by Grothendieck's Existence Theorem

2. Schlessinger

3. GET

4. Take a family of curves $\mathscr{C} \to \operatorname{Spec} A \to \operatorname{Spec} A'$ with $M = \ker(A' \to A)$. Then

$$\mathscr{O}_{\mathscr{C}}(M) = H^2(\mathscr{C}_{A_0}, T_{\mathscr{C}_{A_0}/A_0} \otimes M)$$

, $D_{a_0}(M) = H^1(\mathscr{C}_{A_0}, T_{\mathscr{C}_{A_0}/A_0} \otimes M)$ and A_0-modules. We note that this is all compatible with etale base change $A_0 \to B_0$ (Hartshorne), completion (Hartshorne) and cohomology and base change (H). So looking at $\mathscr{C}_0 \xrightarrow{f} \operatorname{Spec} A_0$. We want $R^i f_*(T_{\mathscr{C}_{A_0}/A_0} \otimes M) \otimes k(p) \to H^i(\mathscr{C}_p, T_{C_p/p})$ to be isomorphisms.

Note that there are no nontrivial infinitesimal automorphisms $H^0(\mathscr{C}, T)$, then we have a DM stack.

Theorem 2 (Artin)**.** *$\mathscr{F}$ is an Artin stack locally of finite type over S if*

1. *S1,S2 hold and if $a_0 \in \mathscr{F}(A_0)$ and M is a finite A_0-module, then $\operatorname{Aut}^{inf}_{a_0}(A_0[M])$ is a finite A_0-module.*

2. *$\mathscr{F}(\hat{A}) \to \varprojlim \mathscr{F}(\hat{A}/\mathfrak{m}^n)$ equivalence*

3. *$D, \mathscr{O}, \operatorname{Aut}^{inf}_{a_0}(A_0[M])$ satisfy (4.1)*

4. *If φ is an automorphism of a_0 such that $\varphi = \mathrm{id}$ at a dense set of points of $\operatorname{Spec} A_0$, then $\varphi = \mathrm{id}$.*

5. *Because 1-4 imply that $\mathscr{F} \to \mathscr{F} \times \mathscr{F}$ is representable and separated, it makes sense to require that it is quasi-compact.*

2 Olsson

Let $f : X \to S$ be a morphism of schemes and set

$$\begin{array}{ccc} GF(\mathscr{O}_X) = f^{-1}\mathscr{O}_X\{F(\mathscr{O}_X)\} & \xrightarrow{\pi} & \mathscr{O}_X \\ \uparrow & \nearrow & \\ f^{-1}\mathscr{O}_S & & \end{array}$$

F is a functor from $f^{-1}\mathscr{O}_S$ algebras to sheaves of sets, and G is a functor in the other direction.
$\tau_{\geq -1}|_{X/S} = (I/I^2 \to \Omega^1_{GF(\mathscr{O}_X)/f^{-1}\mathscr{O}_S})$ with $I = \ker \pi$.

Theorem 3. *For any quasicoherent $\mathscr{O}_X$-module M, $ch(\tau_{\leq 0}(R\mathscr{H}om(\tau_{\geq -1}L_{X/S}, M)[1])) \simeq \operatorname{Exal}_S(X, M)$*

Where $\tau_{\geq -1}L_{X/S}$ is the truncated cotangent complex.
$L_{X/S}$ is the full cotangent complex.
Given $n \geq 0$, $GH \dots GFGF(\mathscr{O}_X)$ $n+1$ times, which is an $f^{-1}\mathscr{O}_S$-algebra, $\mathscr{A}_n \to \mathscr{O}_X$ is surjective from the adjunction.

$\mathscr{A}_*$ is a simplicial $f^{-1}\mathscr{O}_S$-algebra, take $a : GF \to \mathrm{id}$, and $b : \mathrm{id} \to FG$, and we get $d_i : \mathscr{A}_{n+1} \to \mathscr{A}_n$ and $s_d : \mathscr{A}_n \to \mathscr{A}_{n+1}$.

$\mathscr{A}_2 = GFGFGF(\mathscr{O}_X) \to GFGF(\mathscr{O}_X) = \mathscr{A}_1$. We can get 3 maps down by taking natural transformations of $GF \to \mathrm{id}$

$\tilde{L}_*$ is $(\Omega_{\mathscr{A}_*}/f^{-1}\mathscr{O}_S) \otimes \mathscr{O}_X$ and it has $n+1$ maps down at each point. These are simplicial $\mathscr{O}_X$-modules. So we define the cotangent complex to be $L_{X/S}$, with $\tilde{L}_2 \overset{d_0-d_1+d_2}{\to} \tilde{L}_1 \overset{d_0-d_1}{\to} \tilde{L}_0$.

Remark 1. *This is an actual complex of flat $\mathscr{O}_X$-modules.*

1. $\mathscr{H}^i(L_{X/S})$ is quasicoherent and coherent if S is locally noetherian and f is of finite type.

2. Call the following diagram $(*)$
$$\begin{array}{ccc} X' & \xrightarrow{u} & X \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\ Y' & \xrightarrow{v} & Y \end{array}$$
then there is a base change morphism $u^*L_{X/Y} \to L_{X'/Y'}$. If $(*)$ is cartesian and tor independent, then this map is a quasi-isomorphism and $f'^*L_{Y'/Y} \oplus u^*L_{X/Y} \to L_{X'/Y}$ is a quasi-isomorphism.

3. $X \overset{f}{\to} Y \overset{g}{\to} Z$ then there is a distinguished triangle $f^*L_{Y/Z} \to L_{X/Z} \to L_{X/Y} \to f^*L_{Y/Z}[1]$.

4. $\tau_{\geq -1}L_{X/Y}$ is equal to our earlier defined object with the same notation.

Remark 2. *1. $\mathscr{H}om(L_{X/Y}) = \Omega^1_{X/Y}$*

2. If f is smooth, then $L_{X/Y} \to \Omega^1_{X/Y}$ is a quasi-isomorphism

3. If $X \hookrightarrow Y$ is a closed immersion which is locally a complete intersection, then $L_{X/Y} = I/I^2[1]$.

Theorem 4 (Illusie). *$ch(\tau_{\geq -1}(R\mathscr{H}om(L_{X/Y}, I)[1])) \simeq \mathrm{Exal}_Y(X, I)$ implies that we have $\mathrm{Ext}^1(L_{X/Y}, I) \simeq \mathrm{Exal}_Y(X, I)$, $\mathrm{Ext}^0(L_{X/Y}, I) = \hom(\Omega^1_{X/Y}, I)$ is the automorphism group of any $X \hookrightarrow X'$ by I as a Y-morphism.*

Problem:

$$\begin{array}{ccc} X_0 & \overset{i}{\dashrightarrow} & X \\ \downarrow{\scriptstyle f_0} & & \vdots{\scriptstyle f} \\ Y_0 & \overset{j}{\hookrightarrow} & Y \\ \downarrow & \swarrow & \\ S & & \end{array}$$

With j a closed immersion defined by a square zero ideal J. Fill in the diagram as indicated with i square-zero such that $f_0^*J \simeq \ker(\mathscr{O}_X \to \mathscr{O}_{X_0})$.

Solution:

$X_0 \to Y_0 \to Y$ gives $f_0^*L_{Y_0/Y} \to L_{X_0/Y} \to L_{X_0/Y_0} \to f_0^*L_{Y_0/Y}[1]$, and so we get a long exact sequence

$$0 \to \mathrm{Ext}^0(L_{X_0/Y_0}, f_0^*J) \to \mathrm{Ext}^0(L_{X_0/Y}, f_0^*J) \to \mathrm{Ext}^0(f_0^*L_{Y_0/Y}, f_0^*J) \to \mathrm{Ext}^1(L_{X_0/Y_0}, f_0^*J) \to$$
$$\mathrm{Ext}^1(L_{X_0/Y}, f_0^*J) \to \mathrm{Ext}^1(f_0^*L_{Y_0/Y}, f_0^*J) \overset{\partial}{\to} \mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J) \to \dots$$

We get that $\mathrm{Ext}^0(f_0^*L_{Y_0/Y}, f_0^*J) = 0$, so we get

$$0 \to \mathrm{Ext}^0(L_{X_0/Y_0}, f_0^*J) \to \mathrm{Ext}^0(L_{X_0/Y}, f_0^*J) \to 0 \to \mathrm{Ext}^1(L_{X_0/Y_0}, f_0^*J) \to \mathrm{Exal}_Y(X_0, f_0^*J) \to$$

$$\hom(f_0^*J, f_0^*J) \xrightarrow{\partial} \mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J) \to \dots$$

Theorem 5. *1. There exists an obstruction $o(f_0) = \partial(\mathrm{id}) \in \mathrm{Ext}^2(L_{X_0/Y_0}, f_0^*J)$ whose vanishing is necessary and sufficient for a solution to the problem*

2. If $o(f_0) = 0$, then the set of isomorphism classes of solution form a torsor under $\mathrm{Ext}^1(L_{X_0/Y_0}, f_0^*J)$

3. $\mathrm{Aut} = \mathrm{Ext}^0(L_{X_0/Y_0}, f_0^*J)$.

Problem:

$$\begin{array}{ccc} X_0 & \overset{i}{\hookrightarrow} & X \\ \downarrow{\scriptstyle f_0} & & \downarrow{\scriptstyle f} \\ Y_0 & \overset{j}{\hookrightarrow} & Y \\ \downarrow{\scriptstyle g_0} & & \downarrow{\scriptstyle g} \\ Z_0 & \overset{k}{\hookrightarrow} & Z \end{array}$$

With maps $h_0 : X_0 \to Z_0$ and $h : X \to Z$ making everything commute, and each horizontal map a small thickening (I for X, J for Y, K for Z) and $g^*K \simeq J$

Theorem 6 (Illusie)**.** *There is a canonical class $o(f_0) \in \mathrm{Ext}^1(f_0^*L_{Y_0/Z_0}, I)$ such that f exists iff $o(f_0) = 0$. If $i(f_0) = 0$ then the set of maps f is a torsor under $\mathrm{Ext}^0(f_0^*L_{Y_0/Z_0}, I)$.*

Sketch: $e(X) \in \mathrm{Ext}^1_{\mathscr{O}_{X_0}}(L_{X_0/Z}, I)$ and $e(Y) \in \mathrm{Ext}^1_{\mathscr{O}_{Y_0}}(L_{Y_0/Z}, J)$ and we have a map $\mathrm{Ext}^1_{\mathscr{O}_{X_0}}(L_{X_0/Z}, I) \to \mathrm{Ext}^1_{\mathscr{O}_{X_0}}(f_0 6 * L_{Y_0/Z}, I)$ by $e(X) \mapsto z_X$.

$$\begin{array}{ccc} \mathrm{Ext}^1_{\mathscr{O}_{Y_0}}(L_{Y_0/Z}, J) & \longrightarrow & \mathrm{Ext}^1_{\mathscr{O}_{X_0}}(f_0^*L_{Y_0/Z}, f_0^*J) \\ & \searrow & \downarrow \\ & & \mathrm{Ext}^1_{\mathscr{O}_{X_0}}(f_0^*L_{Y_0/Z}, I \end{array}$$

With the composition being $e(Y) \mapsto z_Y$.

We want $z_X = z_Y$.

$h_0^*L_{Z_0/Z} \to f_0^*L_{Y_0/Z} \to f_0^*L_{Y_0/Z_0} \to \dots$ and this gives an exact sequence

$$\mathrm{Ext}^0(h_0^*L_{Z_0/Z}, I) \to \mathrm{Ext}^1(f_0^*L_{Y_0/Z_0}, I) \to \mathrm{Ext}^1(f_0^*L_{Y_0/Z}, I) \to \mathrm{Ext}^1(h_0^*L_{Z_0/Z}, I)$$

and the first term is 0, the last term is $\hom(h_0^*K, I)$. The difference of the maps goes to zero in hom, and so must come from an element of the second term, and so that's how we get $o(f_0) = z_X - z_Y$.

3 Osserman

Groupoid Perspective

One nice property: when working with categories fibered in groupoids, we can restrict naturally from global to local and get the right results (eg, we can specify pairs (X_A, φ) with X_A flat over A, $\varphi : X \to X_A$ inducing $X \xrightarrow{\simeq} X_A \otimes_A k$).

Definition 1 (Category Cofibered in Groupoids)**.** *A category cofibered in groupoids over $\mathcal{C}$ is a category fibered in groupoids over $\mathcal{C}^\circ$.*

Definition 2 (Trivial Groupoid)**.** *A groupoid is trivial if there exists exactly one morphism from any object to any other.*

We can refer to "the trivial groupoid" because any trivial groupoid is equivalent to the one with a single object and a single morphism.

Remark 3. *Artin uses S1, Rim uses "homogeneous groupoids."*

Definition 3 (Deformation Stack)**.** *A category $\mathcal{S}$ cofibered in groupoids over $Art(\Lambda, k)$ is a deformation stack if $\mathcal{S}_k$ is trivial and for all $A' \to A, A'' \to A$ with the second surjective, we have*

1. *$\forall \eta_1, \eta_2 \in S_{A' \times_A A''}$, the natural map*

$$Mor_{A' \times_A A''}(\eta_1, \eta_2) \to Mor_{A'}(\eta_1|_{A'}, \eta_2|_{A'}) \times_{Mor_A(\eta_1|_A, \eta_2|_A)} Mor_{A''}(\eta_1|_{A''}, \eta_2|_{A''})$$

   is a bijection

2. *Given $\eta' \in S_{A'}$ and $\eta'' \in S_{A''}$ and $\varphi : \eta'|_A \to \eta''|_{A''}$, there exists $\mathscr{I} \in S_{A' \times_A A''}$ inducing η', η'', φ on restriction.*

Given $\mathcal{S}$, we write $F_{\mathcal{S}} : Art(\Lambda, k) \to Set$ for the functor of isomorphism classes.

Proposition 2. *Let $\mathcal{S}$ be a deformation stack, then $F_{\mathcal{S}}$ is a deformation functor.*

Proof. $F_S(k)$ is the one element set because S_k is trivial.

(H1) follows from 2, and (H2) follows from 1. In fact, get injectivity of (*) as long as $A = k$, since $Mor_A(\eta_1|_A, \eta_2|_A)$ has exactly one element. ☐

Remark 4. *Although being a deformation stack is formally a stronger condition than satisfying (H1) and (H2), it seems that in practice, that any proof of (H1) and (H2) is really a proof of the deformation stack. eg,* Def_X*, earlier proposition actually proves the deformation stack conditions.*

Lemma 1. *If $\mathcal{S}$ is a local deformation problem at a point of an algebraic stack, then $\mathcal{S}$ is a deformation stack.*

Remark 5. *The argument for the lemma directly involves the asymmetry of only $A'' \to A$ being surjective, because we have to use the formal criterion for smoothness applied to the smooth cover by a scheme.*

Lemma 1.4.4 of a paper by Olsson called "Crystalline Cohomology of Stacks and Hyodo-Kato Cohomology."

More good properties of deformation stacks:

1. $A' \to A$, $\ker = I$, $\eta \in S_A$, $\{(\eta', \varphi) | \eta' \in S_{A'}, \varphi : \eta'|_A \eta\} / \simeq$ is a pseudotorsor over $T_S = T_{F_S} \otimes I$.
2. $A' \to A$, $\eta' \in S_{A'}$, $\varphi \in \mathrm{Aut}(\eta'|_A)$, $\{\varphi' \in \mathrm{Aut}(\eta') | \varphi'|_A = \varphi\}$ is a torsor over $\mathrm{Aut}(\zeta_\epsilon) \otimes I$, where ζ_ϵ is the trivial deformation over $k[\epsilon]$.

Proposition 3. *If $\mathscr{S}$ is a deformation stack, then $F_{\mathscr{S}}$ satisfies (H4) iff for $A' \to A$ surjecitve and all $\eta' \in S_{A'}$, the map $\mathrm{Aut}(\eta') \to \mathrm{Aut}(\eta'|_A)$ is surjective.*

In fancier language, in a global setting, (H4) iff the Isom functor is smooth at the identity.

Why deformation "stack"?

Why all these ring fiber products?

$$\begin{array}{ccc} A'' & \longrightarrow & A' \times_A A'' \\ \downarrow & & \downarrow{\scriptstyle surj} \\ A' & \longrightarrow & A \end{array}$$

Lemma 2. *are in bijection with*

$$\begin{array}{ccc} B' & \xrightarrow{q'} & B \\ \downarrow{\scriptstyle q''} & & \downarrow \\ B'' & \longrightarrow & B' \otimes_B B'' \end{array}$$

with $B \hookrightarrow B' \times B''$ *and* $q'(\ker q'')$ *is an ideal.*

$B \hookrightarrow B' \times B''$ iff $\operatorname{Spec} B' \coprod \operatorname{Spec} B'' \to \operatorname{Spec} B$ is scheme theoretically surjective.

The tensor product corresponds to fiber product of schemes, ie, intersections from the point of view of descent theory.