1 Lieblich

Algebraic Stacks

Question: What is geometry?

Answer: Local structure on top of the topology

Example 1. *Let F be a sheaf on S_{fppf}.*

Claim: F is a scheme iff there exists a scheme U and a map $a : U \to F$ which is Zariski-locally an isomorphism.

That is, there exists a covering $\{G_i \subset F\}$ open subfunctor such that for each i, there exists $U_i \subset U$ open with $U_i \simeq G_i$ and each has a map to F.

a is a uniformization.

Definition 1 (Étale Algebraic Space (temp))**.** *An étale algebraic space over S s a sheaf F on S_{ET} such that $\exists$ a scheme U and a surjective étale representable morphism $U \to F$.*

+ hypotheses

You can also define an fppf algebraic space to be the same except that $U \to F$ is only fppf and looking over S_{fppf}.

Theorem 1 (Artin)**.** *Any fppf algebraic space is an étale algebraic space (same hypotheses needed)*

Hypotheses: F is locally of finite presentation over S, $F \to F \times F$ is representable and of finite type (implies q-affine)

Example 2. *$\exists$ a smooth 3-fold over $\mathbb{C}$ with descent datum with respect to* $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ *which is not effective...but there exists a sheaf $\bar{T}/\mathbb{R}$ such that $\bar{T} \otimes \mathbb{C} \simeq T$ and $T \to \bar{T}$ is finite and étale.*

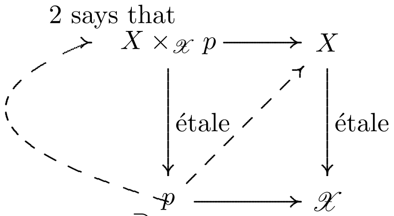
Definition 2 (Deligne-Mumford Stack)**.** *A stack $\mathscr{X}$ on S_{ET} is a Deligne-Mumford Stack if*

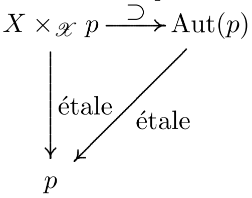
1. *$\mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable by schemes, quasi-compact and separated.*
2. *$\exists$ an étale surjection $X \to \mathscr{X}$ where X is a scheme.*

Note that 1 implies that any map from a scheme to $\mathscr{X}$ is representable.

1 says that for all $f : T \to \mathscr{X}$ and $g : T' \to \mathscr{X}$, $\operatorname{Isom}(pr_1^* f, pr_2^* g) \to T \times T'$ is a quasicompact and separated map of schemes.

2 says that





The second diagram is iff there are no nontrivial infinitesimal automorphism of the object parametrized by p.

DIAGONALS ARE OF FINITE TYPE FOR THIS LECTURE.

So what about $* \to B\mathbb{G}_m$?

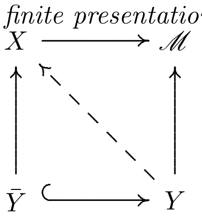
Definition 3 (Artin Stack)**.** *An Artin stack on S_{ET} is a stack $\mathscr{X}$ such that*

1. *$\mathscr{X} \to \mathscr{X} \times_S \mathscr{X}$ is representable by algebraic spaces, quasi-compact and separated*

2. *there exists a scheme X and a smooth surjection $X \to \mathscr{X}$*

Theorem 2 (Artin)**.** *If $\mathscr{X} \to \mathscr{X} \times \mathscr{X}$ representable by algebraic spaces, quasicompact and separated, then there exists $X \to \mathscr{X}$ an fppf surjection iff $\mathscr{X}$ is an Artin stack.*

Proposition 1. *Suppose that $\mathscr{M}$ is a moduli stack (locally of finite presentation) such that isomorphism are representable by algebraic spaces, quasicompact and separated, then $\mathscr{M}$ is algebraic iff $\exists X \to \mathscr{M}$ locally of finite presentation which is formally smooth.*



Theorem 3 (Artin)**.** *An Artin stack $\mathscr{X}$ is DM iff $\mathscr{X} \to \mathscr{X} \times \mathscr{X}$ is unramified iff no object has nontrivial infinitesimal automorphism.*

Example 3. *Let $S = \operatorname{Spec} \mathbb{C}$ and $\mathscr{M}_{1,1}$ the stack of elliptic curves.*

Then $(\mathscr{M}_{(1,1)})_T = \{\pi : \mathscr{E} \to T$ with a section σ such that π is proper and smooth and for all fibers of π, $\mathscr{E}_E$, $g(\mathscr{E}_E) = 1\}$

Proposition 2. *$\mathscr{M}_{1,1}$ is a DM stack.*

The condition on the Isoms is not so bad. The saving grace is that there are no nontrivial infinitesimal automorphisms.

The infinitesimal automorphism of (E, p) over $\operatorname{Spec} \bar{k}$ are parametrized by $H^0(E, T_E) = H^0(E, \mathscr{O}_E) \simeq \bar{k} \supset H^0(E, \mathscr{O}_E(-p)) = 0$, and so it is enough to show that $\mathscr{M}_{1,1}$ is an Artin Stack.

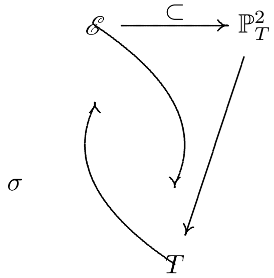
To prove this, find a formally smooth family $B \to \mathscr{M}_{1,1}$, that is, $\mathscr{E} \to B$ with a section σ.

Idea: uniformize by the family of plane cubics

1: There is a scheme U representing the functor mapping T to $C \hookrightarrow \mathbb{P}^2_T$ over T, the smooth families of cubic curves which are étale locally on T, $T' \to T$ such that $C \hookrightarrow \mathbb{P}^2_{T'}$ over T' where C is the vanishing local of a section of $\mathscr{O}_{\mathbb{P}^2_{T'}}(3)$.

Proof of 1: Take the universal cubic $(\sum_{i+j+k=3} \alpha_{ijk} X^i Y^i Z^i) \subset \mathbb{A}^{10} \times \mathbb{P}^2 \to \mathbb{A}^{10}$. Then there exists $\tilde{U} \subset \mathbb{A}^{10}$ parametrizing smooth cubics U =image of $\tilde{U}$ in $\mathbb{P}^9 \leftarrow \mathbb{A}^{10} \setminus \{0\}$.

2: There exists a scheme P ($\to U$)representing the functor T maps to



pointed smooth cubics.

3: Force $\mathscr{O}(1)|_{\mathscr{E}} = \mathscr{O}(3\sigma)|_{\mathscr{E}} \to P'$.

4: Action of $PGL(3)$ on P' coming from choosing coordinates of $\mathbb{P}^2$.

5: $[P'/PGL(3)] \simeq \mathscr{M}_{1,1}$.

2 Olsson

Theorem 4. *$ch : \tilde{C}^{[-1,0]}(T) \to$ (Picard Stacks) is an equivalence of two categories.*

Lemma 1. *$\mathscr{P}$ a Picard stack, then $\exists K^* \in C^{[-1,0]}(T)$ and an equivalence $ch(K) \simeq \mathscr{P}$*

Lemma 2. *$K^*, L^* \in C^{[-1,0]}(T)$ and let $F : ch(K) \to ch(L)$ be a morphism of Picard Stacks. Then there exists a quasi-isomorphism $k : K'^* \to K^*$ and a morphism $\ell : K'^* \to L$ sch that $F \simeq ch(\ell) \circ ch(k)^{-1}$.*

In particular, if $K \in \tilde{C}^{[-1,0]}(T)$ then any morphism $F : ch(K) \to ch(L)$ is isomorphic to $ch(f)$ for some $f : K \to L$.

Sketch of proof: Choose data $\{(U_i, k_u, \ell_i, \sigma_i)\}_{i \in I}$ such that

1. $U_i \subset T$ is open
2. $k_i \in K^0(U_i)$, $\ell_i \in L^0(U_i)$, $\sigma_i : F(k_i) \simeq \ell_i$
3. the map $K'^0 = \oplus_{i \in I} \mathbb{Z}_{U_i} \to K^0$ is surjective.

$K'^{-1} = K^{-1} \times_{K^0} K'^0$.
$\ell : K' \to L$, $\ell^0 : K'^0 \to L^0$ and $\ell^{-1} : K^{-1} \to L^{-1}$. Take $\mathbb{Z}_{U_i} \to L^0$ given by ℓ_i. $(v, (U_i, k_i, \ell_i, \sigma_i)) \in K^{-1}$.
This gives a unique element $t \in L^{-1}$ such that

$$\begin{array}{ccc} F(0) & \xrightarrow{F(v)} & F(k_i) \\ \simeq \uparrow & & \downarrow \sigma_i \\ 0 & \xrightarrow{t} & \ell_i \end{array}$$

The σ_i's define an isomorphism $\sigma : F \simeq ch(\ell) \circ ch(k)^{-1}$

Lemma 3. *$K_1, K_2 \in \tilde{C}^{[-1,0]}(T)$. For two morphism of complexes $f_1, f_2 : K_1 \to K_2$ with associated morphisms $F_1, F_2 : ch(K_1) \to ch(K_2)$ and any isomorphism $H : F_1 \to F_2$, there exists a unique homotopy $h : K_1^0 \to K_2^{-1}$ such that $H = ch(h)$.*

Idea: If $k \in K_1^0$ is section, $H : F_1(k) \to F_2(k)$ corresponds to a section $h(k) \in K_2^{-1}$ such that $dh(k) = f_2(k) - f_1(k)$.

The following definition is preliminary and will soon be replaced:

Definition 4 (Truncated Tangent Complex)**.** *Let $f : X \to S$ be a morphism of schemes. The truncated tangent complex $\tau_{\leq 1} \prod_{X/S}[1] \in \tilde{C}^{[-1,0]}(|X|)$ is the complex with $ch(\tau_{\leq 1} \prod_{X/S}[1]) \simeq \mathrm{Exal}_S(X, \mathscr{O}_X)$.*

Problems

1. This doesn't see $\mathscr{O}_X$-module structure
2. Not the full complex.

Proposition 3. *Let $j : X \hookrightarrow S$ be a closed immersion defined by an ideal I. Then $\tau_{\leq 1} \prod_{X/S}$ is quasi-isomorphic to $\mathscr{N}_{X/S} = \mathscr{H}om(j^* I, \mathscr{O}_X)$*

Proof. $\mathrm{Exal}_S(X, \mathscr{O}_X)$

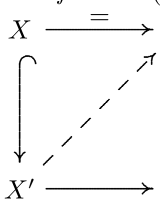
$$\begin{array}{ccc} X & \xhookrightarrow{\mathscr{O}_X} & X' \\ j \downarrow & \swarrow & \\ S & & \end{array}$$

$$\begin{array}{ccccc} \mathscr{O}_X & \xleftarrow{\text{surj}} & \mathscr{O}_{X'} & \hookleftarrow & \mathscr{O}_X \epsilon \\ & \nwarrow^{\text{surj}} & \uparrow & \partial \in \mathscr{N}_{X/S} & \uparrow \\ & & j^{-1}(\mathscr{O}_S / I^2) & \hookleftarrow & j^* I \end{array}$$

□

Proposition 4. *Let $f : X \to S$ be a smooth morphism, then $\tau_{\leq 1} \prod_{X/S}[1] \simeq T_{X/S}[1] = (T_{X/S} \to 0)$*

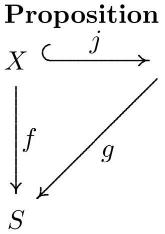
Proof. $shH^0(\tau_{\leq 1} \prod_{X/S}[1]) = 0$.



By $X[\mathscr{O}_X \epsilon]$.

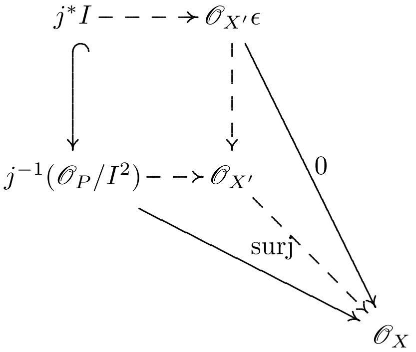
So then $\mathscr{H}^{-1}(\tau_{\leq 1} \prod_{X/S}[1]) = T_{X/S}$. □

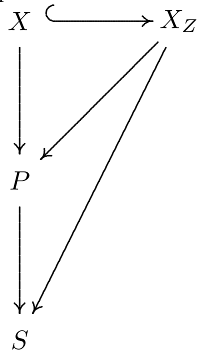
Proposition 5. *Suppose that given a commutative diagram*



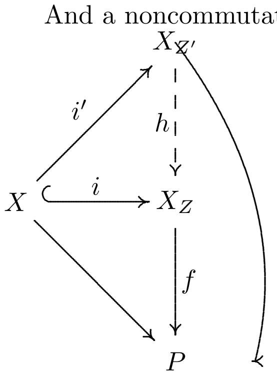
with g smother and j an immersion, then $\tau_{\leq 1} \prod_{X/S}[1] \simeq (j^* T_{P/S} \to \mathscr{N}_{X/P})$

Proof. Let $z : j^* I \to \mathscr{O}_X$ be a section of $\mathscr{N}_{X/P}$.





And a noncommutative diagram



With $z, z' \in \mathscr{N}_{X/P}$ and $(f \circ h - f') \in j^* T_{P/S}$.

The upshot is that there exists a fully faithful functor $pch(j^* T_{P/S} \to \mathscr{N}_{X/P}) \to \mathrm{Exal}_S(X, \mathscr{O}_X)$

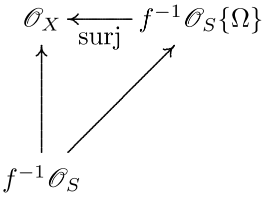
Claim: $ch(j^* T_{P/S} \to \mathscr{N}_{X/P}) \to \mathrm{Exal}_S(X, \mathscr{O}_X)$ is an equivalence. □

So we need a choice of factorization of f and factorization need not exist.

Replacement for factorization: $f : X \to S$, take F : (sheaves of $f^{-1}\mathscr{O}_S$-algebras) $\to$ {sheaves of sets} the forgetful functor, then F has left adjoint $\Omega \mapsto f^{-1}\mathscr{O}_S\{\Omega\}$.

Idea: If $X = \operatorname{Spec} A$ and $S = \operatorname{Spec} B$, then choose elements $f_i \in A$ such that $B[x_i] \to A$ is a morphism of B-algebras.

Now $f^{-1}\mathscr{O}_S\{\Omega\}(V)$ is $f^{-1}\mathscr{O}_S(V)[x_i]_{i\in\Omega(V)}$.



How to choose Ω? Wand $\Omega = F(\mathscr{O}_X)$ and chose open sets $U_i \subset |X|$ and sections $f_i \in \mathscr{O}_X(U_i)$, $\Omega = \coprod_i j_!\{*\}$.

Definition 5 (Truncated Tangent Complex)**.** *The truncated tangent complex of f is the complex of $\mathscr{O}_X$-modules of f is the complex $\mathscr{H}om(\Omega^1_{f^{-1}\mathscr{O}_S\{F(\mathscr{O}_X)\}/f^{-1}\mathscr{O}_S}, \mathscr{O}_X) \to \mathscr{H}om(I/I^2, \mathscr{O}_X)$*

3 Osserman

Effectivity and Algebraization

Two remaining questions:

Question 1 (Effectivity) Suppose that F is a deformation functor coming from a global problem, $R \in \hat{Art}(\Lambda, k)$ and $a \in \hat{F}(R)$ when does η come from a family $\operatorname{Spec} R$ for the original problem?

Question 2 (Algebraization) In same situation with the answer to the above being yes, so we have something over $\operatorname{Spec} R$, when is this induced from an algebraic object, eg, from something over R' of finite type over the base.

Effectivity: No general positive answer. The main tool for positive results is Grothendieck's Existence Theorem:

Theorem 5 (Grothendieck's Existence Theorem)**.** *$f : X \to \operatorname{Spec} A$ proper, A a complete local noetherian ring. Let $A_n = A/\mathfrak{m}_A^{n+1}$ and $X_n = X \otimes_A A_n$. Given $\{\mathscr{F}_n\}$ a compatible collection of coherent sheaves on the X_n, there exists $\mathscr{F}$ on X coherent with $\mathscr{F}|_{X_n} \simeq \mathscr{F}_n$ for all n*

This gives a positive for effectivity in the case of coherent sheaves on a proper scheme.

What about moduli of abstract schemes? It's ok for curves, but fails for K3 surfaces ($K_X = 0$ and $H^1(X, \mathscr{O}_X) = 0$) in this case. If we look at Def_X, it looks like we have a 20 dimensional moduli space. In fact, we have a 20 dimensional space of analytic K3 surfaces, but they are not all algebraic (the algebraic locus is a countable union of 19 dimensional subspaces)

Patch: Work with the moduli of polarized varieties (ie, with a choice of an ample line bundle). It follows from GET (equivalence of categories) that effectivity is satisfied for moduli of polarized (projective) varieties

Now we move on to algebraization.

Artin considers (uni)versal families and proves a positive result quite generally using earlier approximation methods.

This requires that the base S be of finite type over a field or an excellent Dedekind domain.

Definition 6 (Locally of Finite Presentation)**.** *Let $F : Sch_S \to Set$ be a contravariant functor. We say that F is locally of finite presentation over S if for all filtering projective systems of affine schemes $Z_\lambda \in Sch_S$, we have $\varinjlim F(Z_\lambda) = F(\varprojlim Z_\lambda)$*

Why this? EGA: if $F = h_X$ for some $X \in Sch_S$ then this is equivalent to $X \to S$ being locally of finite presentation.

Notation: F is a deformation functor, (R, ξ) with $\xi \in \hat{F}(R)$, $R \in \hat{Art}(\Lambda, k)$ is smooth over F if the induced map $\bar{h}_R \to F$ is smooth.

Theorem 6. *Suppose $F : Sch_S \to Set$ is locally of finite presentation, and $\eta_0 \in F(k)$, given some $\operatorname{Spec} k \to S$ of finite type, with image $s \in S$, let R be a complete local noetherian $\mathscr{O}_{S,s}$-algebra with residue field k, and suppose we $\xi \in F(R)$ which induces η_0 over k, and with (R, ξ) smooth over the local deformation functor corresponding to η_0, then there exists a scheme X of finite type over S and $x \in X$ a closed point and $\eta \in F(X)$ with an isomorphism $\hat{\mathscr{O}}_{X,x} \simeq R$ such that η maps to $\xi_n \in F(R/\mathfrak{m}_R^{n+1})$ for all n.*

In general, this doesn't imply $\eta \mapsto \xi$, unless ξ is uniquely determined by the ξ_n.

Theorem 7. *In the situation above, if ξ is uniquely determined by the ξ_n, then (X, x, η) is unique up to étale morphisms.*