1 Lieblich

Let S be a scheme and $\mathcal{C} = Sch_S$ the big étale site.

For $\mathbb{P}^n$, there are two competing descriptions.

1: Take $h_{\mathbb{P}^n}(T) = \{\mathscr{O}_T^{n+1} \to \mathscr{L}$ surjective with $\mathscr{L}$ invertible on $T\}/\simeq$.

2: $\mathbb{P}^n$ should be $\mathbb{A}^{n+1} \setminus \{0\}$ modulo $\mathbb{G}_m$ with $\mathbb{G}_m(T) = \Gamma(T, \mathscr{O}_T^*)$, $\mathbb{G}_m = \operatorname{Spec} \mathbb{Z}[t, t^{-1}]$.

If 2 makes sense, then $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a $\mathbb{G}_m$-torsor.

$$\begin{array}{ccc} T & \xrightarrow{\mathbb{G}_m\text{-equivariant}} & \mathbb{A}^n \setminus \{0\} \\ \downarrow & \mathbb{G}_m \text{ Torsor} & \downarrow \\ X & \longrightarrow & \mathbb{P}^n \end{array}$$

For all $t \in T, \alpha \in \mathbb{G}_m$, $f(\alpha t) = \alpha f(t)$.

Proposition 1. *There is a natural equivalence of categories from Relatively Affine X-schemes with $\mathbb{G}_m$-action and $\mathbb{G}_m$-equivariant maps and the opposite category of $\mathbb{Z}$-graded quasicoherent $\mathscr{O}_X$-algebras with graded maps.*

Idea: Given $f : Y \to X$ a $\mathbb{G}_m$ action on $f_*\mathscr{O}_Y$ over X, we get it to break up as a sum of eigensheaves induced by the character $\mathbb{G}_m \to \mathbb{G}_m$ by $t \to t^n$.

Example 1. *Action of $\mathbb{G}_m$ on $\mathbb{A}_X^{n+1} = \operatorname{Spec}_X \mathscr{O}_X[x_1, \dots, x_{n+1}]$ graded by total degree.*

Let $T \to X$ be a $\mathbb{G}_m$-torsor. Then it is relatively affine by descent theory. ($\mathbb{G}_m$ is affine).

Proposition 2. *Given a $\mathbb{G}_m$-torsor $T \to X$ there exists an invertible sheaf $\mathscr{L}$ on X such that $T \simeq \operatorname{Spec}_X \oplus_{i \in \mathbb{Z}} \mathscr{L}^i$, the action corresponds to the natural grading by i.*

Proof. fppf locally on X. $T \simeq \operatorname{Spec}_X(\mathscr{O}_X[x, x^{-1}])$. There is a descent datum given by the graded isomorphism $\mathscr{O}[x, x^{-1}] \simeq \mathscr{O}[x, x^{-1}]$.

Note that for each graded piece, it has the form $x^i\mathscr{O}$.

Etc. □

A $\mathbb{G}_m$ equivariant map $\operatorname{Spec}_X \oplus \mathscr{L}^i = T \to \mathbb{A}^{n+1} \setminus \{0\}$ is contained in $\bar{T} \to \mathbb{A}^{n+1}$ which is $\operatorname{Spec}_X \oplus_{\geq 0} \mathscr{L}^i \to \operatorname{Spec}_X \mathscr{O}_X[x_1, \dots, x_{n+1}]$ which is the same as a graded map $\mathscr{O}_X[x_1, \dots, x_{n+1}] \to \oplus_{i \geq 0} \mathscr{L}^i$, and so it's enough to look at $\mathscr{O}_X^{n+1} \to \mathscr{L}$ surjective.

Conclusion: The functor of points tells us that in fact $\mathbb{A}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ is a $\mathbb{G}_m$-torsor.

Let G be a group scheme and X a scheme with a G-action. We would love to make a quotient X/G such that $X \to X/G$ is a G-torsor.

We will do that by allowing X/G to be a stack.

Definition 1 (Quotient Stack)**.** *The quotient stack $[X/G]$ has as fiber category over Y the following:*

Objects are pairs $(T \to Y, \varphi)$ with $T \to Y$ a G-torsor and $\varphi : T \to X$ a G-equivariant morphism, with the arrows $\psi : T \to T'$ commuting with the maps $T \to Y, T' \to Y$ such that ψ is a G-equivariant isomorphism and $\varphi'\psi = \varphi$.

Note, there exists a natural map $\nu : X \to [X/G]$ by

$$\begin{array}{ccc} T & \xrightarrow{\varphi} & X \\ \downarrow & & \\ X & & \end{array}$$

$$\begin{array}{ccc} G \times X & \xleftarrow{\;a\;} & X \\ & \swarrow_{\text{trivial torsor}} & \\ X & & \end{array}$$

With $h(g, x) \mapsto h \cdot gx$ with the first mapping down to (hg, x), and both map to $hg \cdot x$ because it is an action.

Claim: ν makes X a G-torsor over $[X/G]$.

Proof. Iff

$$\begin{array}{ccc} X & \longrightarrow & Y \times_{[X/G]} X \\ \downarrow & & \downarrow \\ Y & \longrightarrow & [X/G] \end{array}$$

What is a fiber product over a stack?

Definition 2. *Given morphisms of stacks $\alpha : \mathscr{X} \to \mathscr{Z}$ and $\beta : \mathscr{Y} \to \mathscr{Z}$ the fiber product has fiber categories $(\mathscr{X} \times_{\mathscr{Z}} \mathscr{Y})_T = \{(x, y, \varphi) | x \in \mathscr{X}_T, y \in \mathscr{Y}_T, \varphi : \alpha(x) \simeq \beta(y)$ is an isomorphism in $\mathscr{Z}_T\}$.*

Example 2. $(X \times_{[X/G]} Y)_T$.

1. $x \in X(T), y \in Y(T)$, $y : T \to Y$
2. $\nu(x) \in [X/G]$ *corresponds to the pullback of* $X \times X \rightrightarrows X$ *to* $y^*U \to U \to X$ *and* $y^*U \to T$.

So φ takes the diagram

$$\begin{array}{ccccc} G \times T & \longrightarrow & y^*U & \longrightarrow & X \\ \downarrow & \swarrow & & & \\ T & & & & \end{array}$$

to a choice of point $U(T)$. So the isomorphism $(x, y, \varphi) \to (x', y', \varphi')$ are precisely the identity maps because X and Y are schemes, so we get a discrete groupoid.

And so we get a G-torsor. □

Note that if G acted freely, we don't need a stack, but we always get one, no matter how bad the G-action is.

$$\begin{array}{ccccc} & & T & & \\ & \swarrow_{\alpha} & & \searrow^{\beta} & \\ X & & & & Y \\ & \searrow_{f} & & \swarrow_{g} & \\ & & \mathscr{Z} & & \end{array}$$

Example 3. *Then $(X \times_{\mathscr{Z}} Y)_T$ is made up of triples $\alpha \in X(T)$, $\beta \in Y(T)$ and $\varphi : f \circ \alpha \to g \circ \alpha$ an isomorphism and $(f \circ pr_1) \circ (\alpha \times \beta) \simeq (f \circ pr_2) \circ (\alpha \times \beta)$*

So we get a map $X \times_{\mathscr{Z}} Y \to X \times Y$.

And then $\mathrm{Isom}(pr_1^* f, pr_2^* g) = X \times_{\mathscr{Z}} Y \to X \times Y$ is our map.

Definition 3 (Representable Morphism)**.** *A morphism of stacks $\mathscr{X} \to \mathscr{Y}$ is representable if $\forall T \to \mathscr{Y}$, $\mathscr{X}_{\mathscr{Y}} T \to T$ is equivalent to a scheme.*

Example 4. *$[*/G]_T$ is to set of maps $U \to T$, so this is the category of G-torsors, $[*/G] = BG$. So $* \mapsto [*/(\mathbb{Z}/2)]$ is finite étale of degree 2. So we can think of it as half of a point.*

2 Olsson

We're going to talk about Picard Stacks. Reminder of the definition [omitted, see Day 6]

$K^* \in C^{[-1,0]}(T)$ (that is, $K^{-1} \to K^0$)

So we get $pch(K^*)$ by taking $pch(K^*)_U$ to be objects $x \in K^0(U)$ and morphisms $x \to y$ is an element $z \in K^{-1}(U)$ such that $dz = y - x$.

And so we get $ch(K)$.

If $\mathscr{P}$ is a Picard stack, then $\mathrm{HOM}(ch(K), \mathscr{P}) \to \mathrm{HOM}(pch(K), \mathscr{P})$ is an isomorphism.

Remark 1. *$pch(K) \to ch(K)$ is fully faithful*

Remark 2. *$f : K_1^* \to K_2^*$ induces a morphism of Picard stacks $ch(f) : ch(K_1) \to ch(K_2)$.*

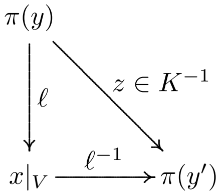
Suppose $f_1, f_2 : K_1^* \to K_2^*$ and a homotopy h between f_1, f_2. (that is, $h : K_1^0 \to K_2^{-1}$ such that $\forall x \in K_1^0$, $f_1(x) - f_2(x) = dh(x)$ and $f_1^{-1} - f_2^{-1} = hd$). Then we get an isomorphism of morphisms $ch(h) : ch(f_1) \to ch(f_2)$.

That is, for all $x \in pch(K_1)$, we get an isomorphism $ch(f_1)(x) \to ch(f_2)(x)$. So for each $x \in K_1^0$ there is a $z \in K_2^{-1}$ such that $dz = f_2(x) - f_1(x)$.

Lemma 1. *If K^{-1} is flasque, then $pch(K)$ is a stack.*

Proof. We have a map $\pi : pch(K) \to ch(K)$, and it is fully faithful. So all we must do is check essential surjectivity. Let $U \subset T$ be open and $x \in ch(K)_U$. Let $\mathscr{L}$ be the sheaf on U which to any any open set $V \subset U$ associates the set of pairs (y, ℓ) with $y \in K^0(V)$ and $\ell : \pi(y) \to x|_V$ in $ch(K)_V$.

Claim: $\mathscr{L}$ is a $K^{-1}|_V$-torsor. The reason is that if we assume that we have $(y', \ell') \in \mathscr{L}$, and



And so $\mathscr{L}$ is classified by an element $[\mathscr{L}] \in H^1(U, K^{-1}|_V) = 0$. □

Observations:

1. The sheaf associated to the presheaf $U \mapsto$ the set of isomorphism classes in $ch(K^*)_U$. So then $\mathscr{H}^0(K^*) = K^0/\operatorname{Im}(K^{-1} \to K^0)$
2. What is the automorphism group of an object $x \in ch(K^*)|_U$? It is $\mathscr{H}^{-1}(K^*)$ because $x \in K^0(U)$ should have $\mathrm{Aut}(x) = \{z \in K^{-1}(U) | dz = x - x = 0\}$.

Corollary 1. *If $f : K_1^* \to K_2^*$ is a quasi-isomorphism, then $ch(f) : ch(K_1) \to ch(K_2)$ is an equivalence.*

Define $\tilde{C}^{[-1,0]}(T) \subset C^{[-1,0]}(T)$ to be the full subcategory of complexes $K^{-1} \to K^0$ with K^{-1} injective.

Theorem 1. *ch induces an equivalence to 2-categories $\tilde{C}^{[-1,0]}(T) \to$ (Picard Stacks over T).*

Corollary 2. *The category of Picard stacks with isomorphism classes of morphisms is equivalent to the category $D^{[-1,0]}(T)$ (Derived Category)*

Lemma 2. *$f : \mathscr{X} \to \mathscr{Y}$ a morphism of stacks and $\bar{f} : X \to Y$ is the corresponding map of sheaves of isomorphism classes. Assume that $\bar{f}$ is an isomorphism and for all $U \subset T$ and $x \in \mathscr{X}_U$ the map of sheaves $\mathrm{Aut}_{\mathscr{X}}(x) \to \mathrm{Aut}_{\mathscr{Y}}(f(x))$ is an isomorphism. Then f is an isomorphism.*

Proof. Given $x, y \in \mathscr{X}_U$ we want $\mathrm{Isom}_{\mathscr{X}}(x,y) \to \mathrm{Isom}_{\mathscr{Y}}(f(x), f(y))$ to be an isomorphism. Injectivity follows from $\alpha, \beta : x \to y$, $f(\alpha) = f(\beta) : f(x) \to f(y)$ then $\alpha^{-1} \circ \beta \in \ker(\mathrm{Aut}_{\mathscr{X}}(x) \to \mathrm{Aut}_{\mathscr{Y}}(f(x))$ implies that $\alpha = \beta$

Surjectivity follows from $\sigma : f(x) \to f(y)$. It is enough to show that σ is in the image locally, so $x, y \mapsto$ the same thing in X. So locally there exists $\tau : x \to y$ such that $\sigma^{-1} \circ f(\tau) : f(x) \to f(x)$.

Essential Surjectivity: $y \in \mathscr{Y}_T$, there exists a covering $T = \cup_i U_i$ and (x_i, ℓ_i) such that $x_i \in \mathscr{X}_{U_i}$ and $\ell_i : f(x_i) \simeq y|_{U_i}$ in $\mathscr{Y}_{U_i}$. Then on U_i there exists a unique isomorphism $\sigma_{ij} : x_i|_{U_{ij}} \to x_j|_{U_{ij}}$ such that the following diagram commutes:

$$\begin{array}{ccc} f(x_i)|_{U_{ij}} & \xrightarrow{f(\sigma_{ij}} & f(x_j)|_{U_{ij}} \\ \downarrow \ell_i & \swarrow \ell_j & \\ y|_{U_{ij}} & & \end{array}$$

With $\sigma_{ij} \circ \sigma_{jk}, \sigma_{ik} : x_i|_{U_{ijk}} \to x_k|_{U_{ijk}}$ are both equal to the unique morphism filling in the top of the above diagram after restricting to U_{ijk}. ☐

Lemma 3. *Let $\mathscr{P}$ be a Picard stack over T. $\{U_i\}$ a collection of open subsets and $k_i \in \mathscr{P}(U_i)$. For all i, $K = \oplus_i \mathbb{Z}_{U_i}$ (where $\mathbb{Z}_{u_i} = j_! \mathbb{Z}$ for $j : U_i \hookrightarrow T$)*

Then there exists a morphism $F : ch(0 \to K) \to \mathscr{P}$ and isoms $\sigma_i ; F(1 \in \mathbb{Z}_{U_i}(U_i)) \simeq k$, and the data $(F, \{\sigma_i\})$ is unique up to unique isomorphism.

Lemma 4. *Let $\mathscr{P}$ be a Picard stack over T. Then there exists $K \in C^{[-1,0]}(T)$ and an isomorphism $ch(K) \simeq \mathscr{P}$.*

Proof. We choose data $\{U_i \subset T\}$ for all $i \in I$ and for all i we choose $k_i \in \mathscr{P}(U_i)$, making these choices such that for all $V \subset T$, $k \in \mathscr{P}_V$ and there exists a cover $V = \cup V_j$ such that $k|_{V_j} = k_i$ for some i with $V_j \subset U_i$. Define $K^0 = \oplus_i \mathbb{Z}_{U_i}$.

So we have $F : ch(0 \to K^0) \to \mathscr{P}$.

Define $K^{-1}(V) = \{(x, \ell), x \in K^0(V), \ell : F(0) \simeq F(x)\}$. We then take the map $K^{-1} \to K^0$ to be $(x, \ell) \mapsto x$ and define $(x, \ell) + (x', \ell') = (x + x', ?)$ where ? is the map $F(0) \simeq F(0) + F(0) \overset{\ell + \ell'}{\to} F(x) + F(x') \simeq F(x + x')$.

So we get a map $pch(K^{-1} \to K^0) \to \mathscr{P}$, and it remains to check equivalence. ☐

Example 5. Pic(X) *the groupoid of line bundles if $ch(\mathscr{O}_X^* \to 0)$*

3 Osserman

<u>Dimensions of Hulls</u>

Mori used a lower bound on dimension of a space of morphisms (in terms of tangent and obstruction spaces) as a key technical tool to prove good theorems about the existence of rational curves on varieties.

Background on obstruction theories:

Definition 4 (Thickening). *Let $\pi : A' \to A$ in $Art(\Lambda, k)$. Then π is a thickening if it is surjective with $\ker \pi \mathfrak{m}_{A'} = 0$. ie, $\ker \pi$ has a k-vector space structure.*

Definition 5 (Obstruction Theory). *Given a predeformation functor F, an obstruction theory for F is a vector space V/k and $\forall \pi : A' \to A$ thickenings, and all $\eta \in F(A)$ an element $ob(\eta, A') \in V \otimes_k \ker \pi$ such that*

1. *$ob(\eta, A') = 0 \iff \exists \eta' \in F(A')$ such that $\eta'|_A = \eta$.*
2. *If $A' \to B \to A$ with $\ker(A' \to A) = I$, $\ker(A' \to B) = J$ then $ob(\eta, B)$ is induced by $ob(\eta, A')$ by $V \otimes I \to V \otimes I/J$.*

Theorem 2. *Suppose F has a hull (R, ξ) and an obstruction theory taking values in V. Then $\dim \Lambda + \dim T_F - \dim V \leq \dim R \leq \dim \Lambda + \dim T_F$..*

If Λ is regular, and the first inequality is an equality, then R is a complete intersection in $\Lambda[[t_1, \ldots, t_r]]$.

Lemma 5. *Suppose that $f : F_1 \to F_2$ is a smooth morphism of predeformation functors and we have an obstruction theory for F_2 taking values in V. Then we obtain an obstruction theory for F_1 taking values in V.*

Proof. Given $A' \to A$, $\eta \in F_1(A)$ set $ob(\eta, A') = ob(f(\eta), A')$. By smoothness, this satisfies (i) and (ii) is a diagram chase. □

Proof of Theorem:

The lemma reduces to the case $F = \bar{h}_R$, since by definition of a hull, $\bar{h}_R \to F$ is smooth and induces an isomorphism $T_R \simeq T_F$.

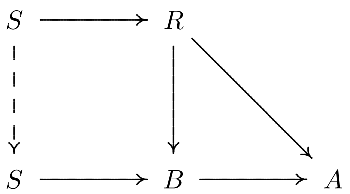
Let $d = \dim T_R$. Schlessinger constructs R as S/J where $S = \Lambda[[t_1, \ldots, t_d]]$, so it is enough to prove that J can be generated by $\leq \dim V$ elements.

By the Artin-Rees lemma, we have $J \cap \mathfrak{m}_S^n \subseteq J\mathfrak{m}_S$ for some n. Set $A' = \Lambda[[t_1, \ldots t_d]]/(\mathfrak{m}_S J + \mathfrak{m}_S^n)$ and $A = \Lambda[[t_1, \ldots, t_d]]/(J + \mathfrak{m}_S^n)$. This gives a thickening $0 \to I \to A' \to A \to 0$ where $I = (J + \mathfrak{m}_S^n)/(\mathfrak{m}_S J + \mathfrak{m}_S^n) = J/\mathfrak{m}_S K$.

We have an object $\xi_A \in \bar{h}_R(A)$ and an obstruction $ob(\xi_A, A')$ to lifting to a map $R \to A'$.

We can write $ob(\xi_A, A') = \sum_{j=1}^{\dim V} v_j \otimes \bar{x}_j$ where the v_j form a basis for V and $\bar{x}_j$ are images of some $x_j \in J$. We want to show that the x_j generate J.

It is enough to see that the $\bar{x}_j$ generate $I = J/\mathfrak{m}_S J$ by Nakayama. Consider $B := A'/(\bar{x}_i)$, this surjects onto A with kernel I'. We get $ob(\xi_A, B) \in V \otimes I'$, but by functoriality, this must be 0, so we have a lift $R \to B$.



So we now want $J \subseteq \mathfrak{m}_S J + (x_i) + \mathfrak{m}_S^n = \ker(S \to B)$

We can choose some $\varphi : S \to S$ making the above commute by choosing $\varphi(t_i)$ appropriately. φ commutes with the two maps to A, so is the identity modulo $J + \mathfrak{m}_S^n$. In particular φ is the identity on $\mathfrak{m}_S/\mathfrak{m}_S^2$, so φ is an isomorphism.

So $\varphi^{-1}(J) \subseteq J + \mathfrak{m}_S^n$ and so we get that $J \subseteq \varphi(J) + \varphi(\mathfrak{m}_S^n) = \varphi(J) + \mathfrak{m}_S^n$. By commutativity of the square, $\varphi(J) \subseteq \mathfrak{m}_S J + (x_i) + \mathfrak{m}_S^n$, and so we have established the theorem.

Example 6. *Say X, Y smooth varieties and let $f : X \to Y$ a morphism. We want to consider* Def_f

Fact: The tangent space is $H^0(X, f^*T_Y)$ and there is an obstruction theory in $H^1(X, f^*T_Y)$. If X is a curve, then $H^0 - H^1$ of f^*T_Y is $\chi(f^*T_Y)$ which is computed by Riemann-Roch.

Example 7 (Deformations of a Smooth Surface)**.** *Tangent space is $H^1(X, T_X)$ and there is an obstruction theory in $H^2(X, T_X)$. If we understand $H^0(X, T_X)$ then we can compute $H^1 - H^2$ of T_X by computing $\chi(T_X)$, and we can use Riemann-Roch For Surfaces to do this.*

eg, if X has finite (discrete) automorphism group in characteristic zero, then $H^0(X, T_X) = 0$.