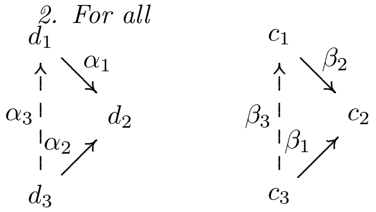
Lieblich

Definition 1 (Category Fibered in Groupoids)**.** *A functor $F : \mathcal{D} \to \mathcal{C}$ is a category fibered in groupoids if*

1. *For all $\beta : c_1 \to c_2$ and for all $d_2 \in \mathcal{D}$ such that $F(d_2) = c_2$, there exists $\alpha : d_1 \to d_2$ such that $F(\alpha) = \beta$.*

2. *For all*



That is, given β_3, there exists a unique α_3 such that $F(\alpha_3) = \beta_3$ and everything commutes.

Definition 2 (Fiber Category)**.** *Given $c \in \mathcal{C}$, the fiber category $\mathcal{D}_c$ has objects $d \in \mathcal{D}$ such that $F(d) = c$ and arrows $\alpha : d_1 \to d_2$ such that $F(\alpha) = \mathrm{id}_c$*

Definition 3 (Morphism of Categories Fibered in Groupoids)**.** *A 1-morphism of categories fibered in groupoids $F_1 : \mathcal{D}_1 \to \mathcal{C}$ and $F_2 : \mathcal{D}_2 \to \mathcal{C}$ is a functor $F : \mathcal{D}_1 \to \mathcal{D}_2$ which commutes with the functors to $\mathcal{C}$.*

F is an equivalence (isomorphism) if $\forall c \in \mathcal{C}$ the induced $F_c : (\mathcal{D}_1)_c \to (\mathcal{D}_2)_c$ is an equivalence.

Note: $\hom(\mathcal{D}_1, \mathcal{D}_2)$ is a groupoid (arrows are natural isomorphisms of functors $\mathcal{D}_1 \to \mathcal{D}_2$)

So now take $\mathcal{C} = Sch_S$. We have our old friend, $Func(\mathcal{C}^\circ, Sets)$ and our even older friends Schemes over S.

We note that our old(er) friends naturally define categories fibered in groupoids.

Example 1. *$\mathcal{D}_1 = h_X$, $X \in Sch_S$. So look at $\hom_{\mathcal{C}}(h_X, \mathcal{D}_2) \xrightarrow{\simeq} (\mathcal{D}_2)_X$ is an equivalence of categories.*

Remember that $\mathscr{M}_0$ is the moduli of varieties (we've been vague here, but it is some object such that every $X \to \mathscr{M}_0$ determines and is determined by a flat family $\mathscr{V} \to X$)

Example 2. *$X \mapsto QCoh(X)$ the category of quasicoherent sheaves on X with isomorphisms as the arrows defines a category fibered in groupoids.*

Bonus: Descent Theory = Gluing = Sheafiness

Gluing in general: Fix $\mathcal{D} \to \mathcal{C} = Sch_S$ thought of as a Site (say, big Étale)

Definition 4 (Category of Descent Data)**.** *Given a covering $\{Y_i \to X\}$ the category of descent data with respect to that covering is $\mathscr{D}_{\{Y_i \to X\}}$ with objects (d_i, φ_{ij}) where $d_i \in \mathcal{D}_{Y_i}$ and $\varphi_{ij} : d_i|_{Y_i \times_X Y_j} \to d_j|_{Y_i \times_X Y_j}$, that is, $pr_1^* d_i \to pr_2^* d_j$, an isomorphism, such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ on $Y_i \times_X Y_j \times_X Y_k$, with the arrows being $(d_i, \varphi_{ij}) \to (d_i', \varphi_{ij}')$ being $d_i \to d_i'$ compatible with $\varphi_{ij}, \varphi_{ij}'$.*

Observation: Any object of $\mathscr{D}_X$ gives rise to an object of $\mathscr{D}_{\{Y_i \to X\}}$ by $d_i = d|_{Y_i} = \psi^*(d)$ and $\psi_i pr_1 = \psi_j pr_2$ so we get a natural isomorphism $pr_1^* \psi_i^* \to pr_2^* \psi_j^*$ and so $pr_1^* d_i \simeq pr_2^* d_j$, so the cocycle condition is built into pseudofunctors.

Upshot: Get a functor $\mathscr{D}_X \to \mathscr{D}_{\{Y_i \to X\}}$.

Definition 5 (Stack)**.** *$\mathcal{D}$ is a prestack on $\mathcal{C}$ if $\nu_{\{Y_i \to X\}}$ is fully faithful for all $\{Y_i \to X\}$ (descent morphisms)*

$\mathcal{D}$ is a stack if $\nu_{\{Y_i \to X\}}$ is an equivalence of categories for all $\{Y_i \to X\}$ (effective descent morphism)

Prestack: A Reinterpretation

Given $a, b \in \mathcal{D}_X$, so now we define a presheaf $I(a, b)$ on Sch_X as follows: given $f : Y \to X$ assign $I(a, b)(f) = \mathrm{Isom}_{\mathcal{D}_Y}(f^* a, f^* b)$.

Lemma 1. *$\mathcal{D}$ is a prestack iff $\forall X, a, b$, $I(a, b)$ is a sheaf on X_{ET}.*

Exercise 1. *Prove this.*

This is that the isomorphisms form a sheaf.

Just as one can sheafify a presheaf, one can stackify a prestack (or in fact, any category fibered in groupoids)

Theorem 1. *Given a fibered category $\mathcal{D} \to \mathcal{C}$ with $\mathcal{C}$ a site, there exists a stack $\mathcal{D}^s$ and a 1-morphism $\mathcal{D} \to \mathcal{D}^s$ over $\mathcal{C}$ such that for all stacks $\mathcal{S} \to \mathcal{C}$, the map $\hom(\mathcal{D}^s, \mathcal{S}) \to \hom(\mathcal{D}, \mathcal{S})$ is an equivalence of groupoids.*

Proposition 1. *QCoh is a stack on $(\operatorname{Spec}\mathbb{Z})_{fppf} = (Sch_{\mathbb{Z}})_{fppf}$.*

Proposition 2. *Sheaves on $(\operatorname{Spec}\mathbb{Z})_{ET}$ form a stack. ($Sh_T = \{$sheaves on $T_{ET})$*

<u>Our Problems</u>: Is it a stack

5: The subspaces of a vector space V: STACK - because they're a Sheaf, and a sheaf is a stack.

4: Closed subschemes of X: STACK - sheaf

3: $\hom(X, Y)$: STACK - sheaf

2: Line bundles on X: STACK, but not a sheaf (fails in as many ways as possible, but it is a stack due to descent theory)

1: Curves of Genus $g = 1$: Stack, but not a sheaf (See Ravi's second talk)

0: Varieties: Prestack ($\operatorname{Isom}(X, Y)$ is a sheaf), but not a stack.

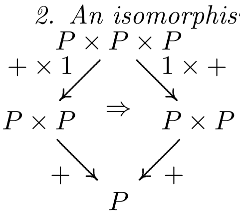
Example 3. *There exists $X/\mathbb{C}$ a smooth 3-fold, with a descent datum relative to $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$ which does NOT descent (so it is not quasi-projective)*

Funny: A scheme X is a sheaf, so a family $X \to Y$ is a sheaf on T_{ET}. So $\{Schemes\} \subset Sheaves$, so why not take the stacky closure of Sch in Sh?

<u>Olsson</u>

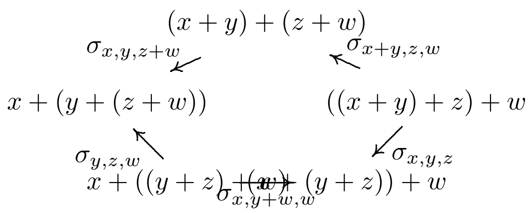
Definition 6 (Picard Category)**.** *A Picard Category is a groupoid $\mathcal{P}$ together with the following extra structure:*

1. *A functor $+ : P \times P \to P$*
2. *An isomorphism of functors:*



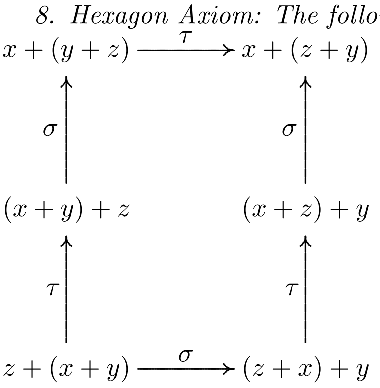
$\sigma_{x,y,z} : (x + y) + z \simeq x + (y + z)$

3. *A natural transformation $\tau_{x,y} : x + y \simeq y + x$ commuting with $+$.*
4. *For all $x \in P$, the functor $P \to P$ by $y \mapsto x + y$ is an equivalence*
5. *Pentagon Axiom: The following diagram commutes*



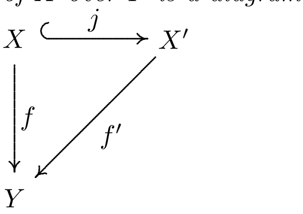
6. *$\tau_{x,x} = \mathrm{id}$ for all $x \in P$*
7. *$\forall x, y \in P$, $\tau_{x,y} \circ \tau_{y,x} = \mathrm{id}$*

8. Hexagon Axiom: The following diagram commutes:



Example 4. *If X is a scheme, then $\mathrm{Pic}(X)$, the groupoid of all line bundles on X with $\otimes : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathrm{Pic}(X)$*

Example 5. *$f : X \to Y$ a morphism of schemes, and I a quasicoherent $\mathscr{O}_X$-module. Then an I-extension of X over Y is a diagram*

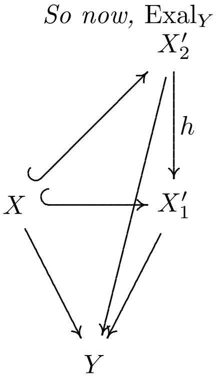


where j is square zero together with an isomorphism $I \xrightarrow{\iota} \ker(\mathscr{O}_{X'} \to \mathscr{O}_X)$. Let $\mathrm{Exal}_Y(X, I)$ to be the category of I-extensions of X over Y.

Remark: $I \to \mathscr{O}_{X'} \to \mathscr{O}_X$ over $f^{-1}\mathscr{O}_Y$

If $A \to B$ is a morphism of sheaves of algebras on a topological space T and I is a B-module, we get the category $\mathrm{Exal}_A(B, I)$ (Extensions of Algebras)

So now, $\mathrm{Exal}_Y(X, I)$ is a groupoid. Take



Such that it commutes with the isomorphisms of I with $\ker(\mathscr{O}_{X'_i} \to \mathscr{O}_X)$, and so we get an isomorphism.

If $U \subseteq X$, then there is a restriction functor $\mathrm{Exal}_Y(X, I) \to \mathrm{Exal}_Y(U, I_U)$

For $u : I \to J$, a maps of $\mathscr{O}_X$-modules, there is a functor $u_* : \mathrm{Exal}_Y(X, I) \to \mathrm{Exal}_Y(X, J)$ so we get $I \hookrightarrow \mathscr{O}_{X'} \to \mathscr{O}_X$ as a morphism of $f^{-1}\mathscr{O}_Y$ algebras. $\mathscr{O}_{X'_u} = \mathscr{O}_{X'} \oplus_I J = (\mathscr{O}_{X'}[J])/\{(i, -u(i)) | i \in I\}$.

So we get a map $X \to X'_u$ from J which commutes with the map $X \to X'$ by I as morphisms over Y.

Lemma 2. *If I and J are two quasi-coherent $\mathscr{O}_X$-modules, then*

$$(pr_{1*}, pr_{2*}) : \mathrm{Exal}_Y(X, I \oplus J) \to \mathrm{Exal}_Y(X, I) \times \mathrm{Exal}_Y(X, J)$$

is an equivalence of categories.

Take $\Sigma : I \times I \to I$ to be the summation map, so we define the $+$ map $+ : \mathrm{Exal}_Y(X, I) \times \mathrm{Exal}_Y(X, I)$ by taking first the isomorphism with $\mathrm{Exal}_Y(X, I \oplus I)$ and then Σ_*. σ, τ are determined by taking more or less the canonical isomorphisms, and then everything works out.

Example 6. *Let $f : A \to B$ be a homomorphisms of abelian groups. Define P_f to have objects the elements $x \in B$ and morphisms $x \to y$ elements $h \in A$ with $f(h) = y - x$*

Let T be a topological space (or a site)

Definition 7 (Picard Stack)**.** *A Picard (pre-)Stack over T is a (pre)Stack $\mathcal{P}$ with morphisms of stacks $(+, \sigma, \tau)$ such that for all $U \subseteq T$, the fiber $(\mathcal{P}_U, +, \sigma, \tau)$ is a Picard category.*

Example 7. $\mathrm{Pic}(-)$ *defines a Picard Stack on $|X|$.*

Example 8. $\mathrm{Exal}_Y(-, I)$ *gives a Picard stack on $|X|$*

Example 9. *$f : A \to B$ is a homomorphism of sheaves of abelian groups on a topological space T, then get Picard prestack $pch(A \to B)$*

Definition 8 (Morphism of Picard Stacks)**.** *Let T be a topological space and $\mathcal{P}_1, \mathcal{P}_2$ Picard Stacks over T. Then a morphism of Picard Stacks $\mathcal{P}_1 \to \mathcal{P}_2$ is a pair (F, ι) where $F : \mathcal{P}_1 \to \mathcal{P}_2$ is a morphism of stacks and $\iota : F(x + y) \simeq F(x) + F(y)$ such that the following commute*

$$
\begin{array}{ccc}
F(y) + F(x) & \xleftarrow[\tau]{} & F(x) + F(y) \\
\uparrow \iota & & \uparrow \iota \\
F(y + x) & \xleftarrow[F(\tau)]{} & F(x + y)
\end{array}
$$

$$
\begin{array}{ccccc}
F((x + y) + z) & \xrightarrow{\iota} & F(x + y) + F(z) & \xrightarrow{\iota} & (F(x) + F(y)) + F(z) \\
F(\sigma) \uparrow & & & & \uparrow \sigma \\
F(x + (y + z)) & \xrightarrow{\iota} & F(x) + F(y + z) & \xrightarrow{\iota} & F(x) + (F(y) + F(z))
\end{array}
$$

We get another Picard stack $\mathrm{HOM}(\mathcal{P}_1, \mathcal{P}_2)$ with an identity element, kernels and $\otimes$.

Osserman

Continuing the Proof of Schlessinger's Criterion.

Already shown that (H1)-(H3) imply that we have a Hull. Suppose that F has a hull (R, ξ). Then $(H3)$ follows from $T_R \simeq T_F$ and R noetherian of finite dimension.

Now suppose that we have $p' : A' \to A$ and $p'' : A'' \to A$ in $Art(\Lambda, k)$ with p'' a surjection. For (H1), we want (*) $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$ to be surjective. Suppose that we have $\eta' \in F(A')$ and $\eta'' \in F(A'')$ both restricting to $\eta \in F(A)$. Since $\bar{h}_R \to F$ is smooth (by exercise) is it surjective, so $\exists u' : R \to A'$ such that $u'(\xi) = \eta'$.

Also, using smoothness applied to p'', $\exists u'' : R \to A''$ with $u''(\xi) = \eta''$. Set $\zeta = u' \times_u u''(\xi) \in F(A' \times_A A'')$, this lifts (η', η'') and this proves (H1).

For (H2), assume that $A = k$, $A'' = k[\epsilon]$. We want (*) injective. Suppose that $v \in F(A' \times_A A'')$ also restricts to η' and η'', we want $v = \zeta$. Keeping the same $u' : R \to A'$, apply smoothness to the map $A' \times_k k[\epsilon] \to A'$ and obtain to obtain $q'' : R \to k[\epsilon]$ such that $u' \times q''(\xi) = v$. Because $T_R \simeq T_F$, and we have $u' \times u''(\xi) = \zeta$, we have $u'', q'' \in T_R$ so since $u''(\xi) = \zeta|_{A''} = v|_{A''} = q''(\xi)$ so $u'' = q''$ so $\zeta = v$. This is (H2), and so done.

So now assume that (H1)-(H4) are satisfied. We already have a hull (R, ξ). We want to show that it prorepresents F. That is, for all Artin rings A, we have a bijection $h_R(A) \to F(A)$. It's always a surjection by smoothness, so injectivity must be checked.

We prove this by induction on the length of A. Let $p' : A' \to A$ be a small thickening and let I be the kernel. Suppose $h_R(A) \to F(A)$ is a bijection. We want to deduce that we have a bijection for A' as well. For all $\eta \in F(A)$, we have $h_R(p)^{-1}(\eta)$ and $F(p)^{-1}(\eta)$ (the -1's denote inverse images, not inverse functions), and both are pseudotorsors under $T_F \otimes I \simeq T_R \otimes I$. By functoriality, they are compatible.

But we have injection, so they must be in bijection. Since this holds for all $\eta \in F(A)$, we have a bijection $h_R(A') \simeq F(A')$. So (R, ξ) prorepresents F, by induction.

If F is prorepresentable, then (*) is always bijective because $A' \times_A A''$ is a categorical fiber product in $\hat{Art}(\Lambda, k)$.

And so Schlessinger's Criterion is established.

More Examples:

Example 10 (Deformations of a quotient scheme)**.** *Let X_Λ be a scheme over Λ and $\mathscr{E}_\Lambda$ a quasicoherent sheaf on X_Λ. Write $X, \mathscr{E}$ for the restrictions to k.*

Fix $\mathscr{E} \to \mathscr{F}$ surjective as a quasicoherent quotient. Then $\mathrm{Def}_{\mathscr{F},\mathscr{E}}$ sends A to $\{\mathscr{E}_\Lambda|_A \to \mathscr{F}_A$ flat restricting to $\mathscr{E} \to \mathscr{F}$ after $\otimes k\}$.

Note: No automorphism to worry about, so we could even have a notion of equality of quotients via equality of kernels.

Theorem 2. *$\mathrm{Def}_{\mathscr{F},\mathscr{E}}$ is a deformation functor, and satisfies (H4). If X_Λ is proper and $\mathscr{E}$ is coherent, then $\mathrm{Def}_{\mathscr{F},\mathscr{E}}$ also satisfies (H3), and so is prorepresentable.*

Note: For representability of the global version (Quot Scheme) need projective. But we see that the local behavior is still scheme-like under properness hypothesis. This hints at algebraic spaces.

Sketch of proof:

Given $A' \to A$, $A'' \to A$ and $\mathscr{F}_{A'}, \mathscr{F}_{A''}$ both restricting to $\mathscr{F}_A$ on A, set $B = A' \times_A A''$ and set $\mathscr{F}_B = \mathscr{F}_{A'} \times_{\mathscr{F}_A} \mathscr{F}_{A''}$ and get a surjection $\mathscr{E}_B := \mathscr{E}_\Lambda|_B \to \mathscr{F}_B$ by $\mathscr{E}_B \to \mathscr{E}_{A'} \times_{\mathscr{E}_A} \mathscr{E}_{A''} \to \mathscr{F}_B$. This is not necessarily an isomorphism.

This gives (H1), but we actually constructed an inverse to (*), so we get (H2) and (H4) also.

The tangent space $\mathrm{Def}_{\mathscr{F},\mathscr{E}}$ is $H^0(X, \mathscr{H}om(\mathscr{G}, \mathscr{F}))$ where $\mathscr{G} = \ker(\mathscr{E} \to \mathscr{F})$. (this is an exercise)

Under our extra hypothesis, this is finite dimensional, and so (H3) is satisfied.

Corollary 1. *Given X_Λ/Λ and $Z \subseteq X$, then $\mathrm{Def}_{Z,X}$ is a deformation functor and satisfies (H4). If further X_Λ is proper over Λ, then (H3) is satisfied and so prorepresentable.*

Proof. Set $\mathscr{E}_\Lambda = \mathscr{O}_{X_\Lambda}$, then closed subschemes are just the quasicoherent quotients of this. Apply the Theorem □

Example 11. *Given X_Λ, Y_Λ over Λ, $f : X \to Y$ over k, and Def_f sends A to $\{f_A : X_\Lambda|_A \to Y_\Lambda|_A$ over A restricting to f on $A\}$.*

Corollary 2. *If X_Λ, Y_Λ are locally of finite type, X_Λ is flat over Λ and Y_Λ separated over A, then Def_f is a deformation functor and satisfies (H4). If X_Λ, Y_Λ are proper, then also get (H3).*