Starr

Deformation Theory and a Theorem of Mori

Let $k = \bar{k}$ be fixed.

Definition 1 (Fano Manifold)**.** *A Fano Manifold X over k is a smooth, proper, connected variety over k such that* $\det(T_X)(= \omega_X^{-1})$ *is ample.*

Example 1. *1. The only curves that are Fano are isomorphic to $\mathbb{P}^1$.*

2. The only surfaces are the Del Pezzo surfaces (like $\mathbb{P}^1 \times \mathbb{P}^1, \mathbb{P}^2$,blowups of $\mathbb{P}^2$ at fewer than 9 points in general position).

3. X_d is a smooth hypersurface in $\mathbb{P}^n$ of degree d is Fano if $d \leq n$.

Theorem 1 (Mori)**.** *Every Fano manifold of* $\dim \geq 1$ *is uniruled.*

Definition 2 (Uniruled)**.** *A variety is uniruled if every closed point p of X is contained in the image of a finite $f : \mathbb{P}^1 \to X$.*

Obstruction Theory

Let R be a local complete noetherian ring with algebraically closed residue field k. Let $\mathcal{C}_R$ be the category of local Artinian R-algebras with residue field κ.

Infinitesimal Extension: $A' \xrightarrow{q} A$ such that $\ker q = N$ with N a finite dimensional k-vector space.

A map of infinitesimal extensions is a collection of vertical maps making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle u_N} & & \downarrow{\scriptstyle u_{A'}} & & \downarrow{\scriptstyle u_A} & & \\ 0 & \longrightarrow & \tilde{N} & \longrightarrow & \tilde{A}' & \longrightarrow & \tilde{A} & \longrightarrow & 0 \end{array}$$

Let $F : \mathcal{C}_R \to Sets$ be a functor with $F(k) = \{pt\}$. Then a deformation situation is $(\Sigma, x \in F(A))$. A morphism of deformation situations is $u : (\sigma, x) \to (\tilde{\Sigma}, \tilde{x})$ with $u : \Sigma \to \tilde{\Sigma}$ such that $u_*(x) = \tilde{x}$.

So Obstruction theory gives a pair $(\mathscr{O}, \omega)$ where $\mathscr{O}$ is a finite dimensional k-vector space $(N \mapsto N \otimes_k \mathscr{O})$ and ω is a rule $(\Sigma, x) \to \omega_{\Sigma,x} \in N \otimes_k \mathscr{O}$ which is

1. Suitably Natural, ie, $\forall u : (\Sigma, x) \to (\tilde{\Sigma}, \tilde{x})$ the image of $\omega_{\Sigma,x}$ under $N \otimes_k \mathscr{O} \to \tilde{N} \otimes_k \mathscr{O}$ is $\omega_{\tilde{\Sigma},\tilde{x}}$. ($\omega$ is the obstruction)

2. $\omega_{\Sigma,x} = 0$ iff x is the image of an element $x' \in F(A')$.

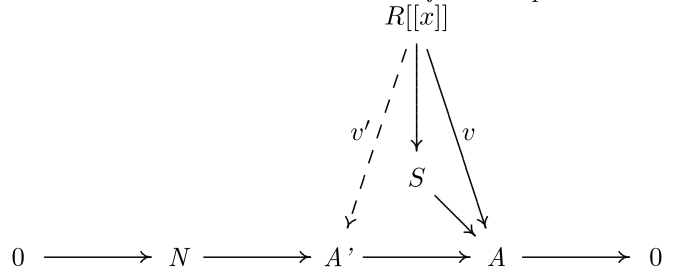
Exercise 1. *(1) implies "if" in (2)*

Example 2. *1. Let $F = h_S$, $S = R[[x]]/I = R[[x_1, \hat{\ldots}, x_r]]/(f_1, \ldots, f_s)$. We take $I/I^2 \to \hat{\Omega}_{R[[x]]/R} \otimes_{R[[x]]} S = R[[x]]\{dx_1, \ldots, dx_r\}$*

$$\hom_{R[[x]]}(\hat{\Omega}_{R[[x]]/R}, K) \to \hom_S(I/I^2, k)$$

by $(\varphi : dx_i \mapsto x_i) \mapsto (f_j \mapsto \sum_{i=1}^r \frac{\partial \bar{f}_j}{\partial x_i} c_i)$.

So then let $\mathscr{O}$ be the cokernel of this map. Take the following diagram



So there exists an R-algebra homomorphism $v' : R[[x]] \to A'$ such that every lift is of the form $v' + \partial$.

$\partial : dx_i \mapsto$ an element of N, so it is a map $\hat{\Omega}_{R[[x]]/R} \to N$. So $v' + \partial$ factors through S iff $f_1, \ldots, f_s \mapsto 0$. Then $v' + \partial$ will give an S-module homomorphism $I/I^2 \to N$ by $f_j \mapsto (v' + \partial)(f_j)$.

Upshot: the element $\omega \in \hom_S(I/I^2, N)/\hom_{R[[x]]}(\hat{\Omega}, N) \simeq (\hom(I/I^2, k)/\hom(\hat{\Omega}, k)) \otimes_k N)$ is independent of the choice of v'. This obstruction vanishes iff w extends to an R-algebra homomorphism $S \to A'$.

2. *Let C be a smooth projection connected curve over k. Let $Z \subseteq C$ be an effective Cartier divisor. Let X be a smooth k-scheme and $f_0 : C \to X$ a morphism. Denote $g = f_0|_Z : Z \to X$ and set $R = k$.*

   Then define a functor F from A to $\{f_A : C \otimes_{\operatorname{Spec} k} \operatorname{Spec} A \to X \times_{\operatorname{Spec} k} \operatorname{Spec} A\}$ so then f_A is a $\operatorname{Spec} A$-morphism such that $f \equiv f_0$ modulo $\mathfrak{m}_A$ and $f|_{Z \times \operatorname{Spec} A} = g \times \operatorname{id}_{\operatorname{Spec} A}$.

   The obstruction theory is $\mathscr{O} = H^1(C, f^* T_X \otimes I_Z)$. So then given a deformation situation $f_A : C_A \to X_A$ and $0 \to N \to A' \to A \to 0$, let $U \subset C$ be an open affine.

$$\begin{array}{ccc} & & X \\ & \nearrow^{f_A|_{U_A}} & \uparrow \tilde{f}_{A,U} \\ U_A & \hookrightarrow & U_{A'} \end{array}$$

   This map is not unique $\mathscr{O}_X \to f_* \mathscr{O}_{U_{A'}}$, but other choices differ by a derivation $\Omega_X \to N \otimes f_* \mathscr{O}_{U_A} \otimes I_Z$.

   Let $\{U_\beta\}$ be an open cover of C. For every β choose $\tilde{f}_{A,U_\beta}$. On $U_\beta \cap U_\gamma$, this makes $\tilde{f}_{A,U_\beta}|_{U_\beta \cap U_\gamma} - \tilde{f}_{A,U_\gamma}|_{U_\beta \cap U_\gamma}$ is an element in $\Omega_X \to N \otimes f_* \mathscr{O}_{U_\beta \cap U_\gamma} \otimes I_Z$. So $\omega_{\Sigma,f} \in \check{H}^1(C_A, \mathscr{H}om_{\mathscr{O}_{C_A}}(f_A^* \Omega_X, N \otimes_k I_{Z_A}))$ and $H^1(C, \mathscr{H}om_{\mathscr{O}_C}(f^* \Omega_X, N \otimes_k I_Z)) = N \otimes_k H^1(C, f^* T_X \otimes I_Z)$.

Fact: Let $F = h_S$ be a prorepresentable functor on C_R. Let $\mathscr{O}_{can} = \hom(I/I^2, k)/\hom(\hat{\Omega}_{R[[x]]/R}, k)$. Let $\mathscr{O}$ be any other obstruction theory. Then there exists a unique $\psi : \mathscr{O}_{can} \to \mathscr{O}$ such that every $\omega_{\Sigma,x}$ is the image of $\Omega_{\Sigma,x,can}$ under ψ.

Sketch the proof:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I/\mathfrak{m}I & \longrightarrow & R[[x]]/L & \longrightarrow & S/\mathfrak{m}_S^c & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow x & & \\ 0 & \longrightarrow & N & \longrightarrow & A' & \longrightarrow & A & \longrightarrow & 0 \end{array}$$

Where $R[[x]]/I + \mathfrak{m}_{R[[x]]}^c = S/\mathfrak{m}_S^c$, $L = \mathfrak{m}_{R[[x]]} I + \mathfrak{m}^c$

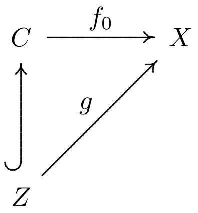
With $\mathfrak{m}_{A'}^c = 0$.

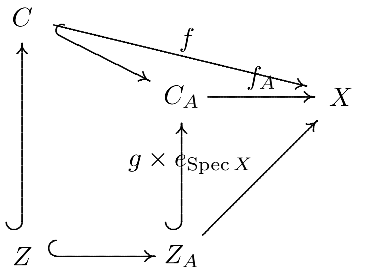
And ψ is injective as it doesn't extend after taking pushouts, so $\omega \in I/\mathfrak{m}_{R[[x]]} I \otimes_k \mathscr{O}$ maps to $0 \in k \otimes_k \mathscr{O}$.

So $I/\mathfrak{m}_{R[[x]]} I$ is a free k-vector space with respect to the basis the images of a minimal set of generators for I.

$S = R[[x_1, \ldots, x_r]]/(f_1, \ldots, f_s)$ and $t_F =$ "dx_1", …, "dx_r".

So as an example, $R = k$ we get $\mathscr{O} = H^1(C, f^* T_X \otimes I_Z)$ and $t_F = H^0(C, f^* T_X \otimes I_Z)$ and the following diagram





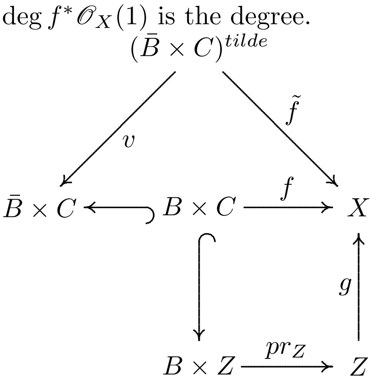
So we get $\dim_k t_F = r = min$ number of generators

$\dim_k \mathscr{O} \geq s$ the minimum number of relations

$\dim S \geq \dim R + r - s \geq \dim R + \dim t_F - \dim \mathscr{O}$ so we get $\dim S/\mathfrak{m}_R S \geq \dim t_F - \dim \mathscr{O}$ and so the second is an equality implies that S is R-flat.

So $\dim S \geq (h^0 - h^1)(C, f^* T_X \otimes I_Z)$ and by R-R, this is $\deg(f^* T_X) + \dim(X)(1 - g(C) - \deg Z)$.

If this dimension is positive, then $\dim_{[f_u]} \hom(C, X; g : Z \to X)$ (X quasiprojective) is positive then $\deg f^* \mathscr{O}_X(1)$ is the degree.



Lemma 1 (Rigidity Lemma)**.** *If $\tilde{f}$ is regular on $\bar{B} \times \{p_i\}$ which is contractible, then $\tilde{f}$ factors through $\bar{B} \times C \overset{pr_C}{\to} C$.*

MISSING: How does this imply the theorem of Mori?

Vakil

Murphy's Law for Deformation Spaces

Hilbert Scheme

$Hilb_n$ will parametrize the closed subschemes of $\mathbb{P}^n$. $Hilb_n = \coprod_{p(t)} Hilb_{n,p(t)}$ where $p(t)$ is the Hilbert polynomial of the projective scheme. And by Hartshorne's Connectedness Theorem, each $Hilb_{n,p(t)}$ is connected. There is also the folklore theorem by Mumford:

Theorem 2 (Murphy's Law)**.** *There is no geometric possibility so horrible that it cannot be found on some Hilbert Scheme.*

It can be found in Harris-Morrison page 18.

Pathologies II - Mumford: Then $\mathbb{P}^3$, degree 14 genus 24 curves, then the Hilbert Scheme is nonreduced!

Definition 3 (Singularity)**.** *A singularity is a pointed scheme (X, p) with an equivalence relation generated by the following: $(X, p) \to (Y, q)$ is a smooth morphism, then $(X, p) \sim (Y, q)$.*

Definition 4. *We say that Murphy's Law holds for a scheme X is every singularity type appears on it.*

Theorem 3. *Murphy's Law holds for the Hilbert Scheme of smooth curves in projective space, of smooth surfaces in $\mathbb{P}^5$, of surfaces in $\mathbb{P}^4$, Kontsevich's space of stable maps, Chow varieties...*

There is a moduli space of smooth surfaces with very ample canonical bundle.

Take $d = 14, g = 24$ and $C \hookrightarrow \mathbb{P}^3$. $X = Bl(\mathbb{P}^3)$. Then $0 \to \operatorname{Aut}\mathbb{P}^3 \to \operatorname{Def}(C \to \mathbb{P}^3) \to \operatorname{Def} X \to 0$. $\operatorname{Def}(C \to \mathbb{P}^3) \to \operatorname{Def} X \to 0? \to \operatorname{ob}(C \to \mathbb{P}^3) \to \operatorname{ob}(X) \to \ldots$. We want a long exact sequence here.

Conjecture 1. *S a surface of general type, then $h^1(\mathscr{O}_X) = 0$ and K_S is ample, then S is unobstructed.*

This is VERY FALSE. Murphy's Law also holds for spaces of surfaces (whether the surfaces are smooth, simply connected, have very ample K, etc)

Plane Curves

Let $C \subset \mathbb{P}^2$ be a curve. We want to deform the curve without changing the singularities. Severi considered the case of deforming curves with only nodes. (He wanted to show that the moduli space of curves is smooth, because every curve can be expressed as a nodal plane curve)

He then proceeded to consider adding cusps so that he could apply his methods to the moduli of surfaces, because a surface is a branched cove of $\mathbb{P}^2$.

Enriques found a flaw in Severi's argument, but tried to fill the gaps, followed by Zariski. Finally, in the 1970s, Wahl proved that Severi was wrong, that there was a curve with a large number of cusps and nodes such that the moduli of surfaces isn't smooth.

In fact, it can be arbitrarily singular.

Philosophy: Anything that isn't as nice as possible can be as bad as possible.

Good Spaces: Curves, Branched Covers of $\mathbb{P}^1$, Surfaces in $\mathbb{P}^3$, Deformations of Nodes

Bad Spaces: Surfaces, Branched Covers of $\mathbb{P}^2$, Surfaces in $\mathbb{P}^4$, Deformations of Cusps

Informal: $S \overset{|2K|}{\to} \mathbb{P}^5$. So there's

$$\begin{array}{ccc} & \operatorname{Def}(S, \mathscr{L}, 6 \text{ sections}) & \\ \swarrow^{\mathbb{C}^*} & \downarrow & \\ \operatorname{Def}(S \hookrightarrow \mathbb{P}^5) & \operatorname{Def}(S, \mathscr{L} = K) & \\ & \downarrow & \\ & \operatorname{Def} S & \end{array}$$

$h^1(\mathscr{O}) = 0$.

We call the embedding π, and take the exact sequence $0 \to T_S \to \pi^* T_{\mathbb{P}^5} \to N \to 0$ and $\operatorname{Def}(S \to \mathbb{P}^5)$ is $H^0(N)$, $\operatorname{Def}(S) = H^1(T_S)$, $Ob(S \to \mathbb{P}^5) = H^1(N)$ and $Ob(S) = H^2(T_S)$. So then if you can put S into $\mathbb{P}^5$ sufficiently positively, $H^1(\pi^* T_{\mathbb{P}^5}) = 0$. So if we know that the moduli of surfaces isn't smooth, we get that smooth surfaces in $\mathbb{P}^5$ is bad, that surfaces in $\mathbb{P}^4$ is bad, etc.

So all we must do is prove that the moduli space of smooth surfaces is bad.

Theorem 4 (Mrev's Universality Theorem)**.** *Fix m, n and define the incidence scheme $\subset (\mathbb{P}^2)^m \times (\mathbb{P}^{2^*})^n$, and we require $p_1 \in \ell_2$, $p_2 \in \ell_3$, etc.*

The Union of the Incidence Schemes satisfies Murphy's Law.

Fact: Fix a line ℓ in $\mathbb{P}^2$ and fix three points on ℓ call them $0, 1, \infty$. Denote each point on ℓ by numbers, and if x, y, z are are points on ℓ, there is a configuration of lines that forces the equation $x + y = z$, there is another that forces $xy = z$ an a third that forces $x = -y$. With these, we can encode any algebraic equation.

"$\mathbb{C}$ cares about $\mathbb{Q}$". What does this mean? Look at the moduli space of $\mathbb{C}$ surfaces

$$\begin{array}{ccc} \mathscr{M} & \longleftarrow & \mathscr{M}_{\mathbb{C}} \\ \downarrow & & \downarrow \\ \operatorname{Spec}\mathbb{Z} & \longleftarrow & \operatorname{Spec}\mathbb{C} \end{array}$$

The singularity type of $yx(y-x)(y-\pi x)$ cannot occur, because π is not algebraic.

Question: Suppose that you have a "nice" object X over $\mathbb{C}$. Is $\operatorname{Def} X$ defined over $\mathbb{Z}$?

This question is open and resolving it either way would be interesting.