Lieblich

We will now return to Moduli problems. Recall:

1. Curves of genus g
2. Line Bundles on X
3. Maps between X and Y
4. Closed Subschemes of X (often $X = \mathbb{P}^n$)
5. Subspaces of a vector space

We even wrote down what we expected the functors of points to be. So do we get Sheaves?

Look at $h_{\mathscr{M}_3}(T) = \hom_T(X_T, Y_T) = \hom(X \times T, Y)$. We proved that Y is a sheaf implies that $h_{\mathscr{M}_3}$ is an fppf sheaf.

How about $h_{\mathscr{M}_4}(T) = \{Z \hookrightarrow X \times T \text{ closed}, Z \text{ is } T\text{-flat}\}/\simeq$ (note, isomorphisms are unique if they exist).

$Z \hookrightarrow X \times T$ is equivalent to $\mathscr{I}_Z \subset \mathscr{O}_{X\times T}$, so the sheaf condition corresponds to descent data on the inclusion $\mathscr{I}_Z \subset \mathscr{O}_{X\times T}$. fppf descent is effective for quasi-coherent sheaves, so these thing glue, so $h_{\mathscr{M}_4}$ is a sheaf.

$\{T_i \to T\}$ gives $h_{\mathscr{M}_4}(T) \to \prod h_{\mathscr{M}_4}(T_i) \rightrightarrows \prod h_{\mathscr{M}_4}(T_i \times_T T_j)$ the products are of isomorphism classes. The uniqueness of isomorphisms tells us that it is harmless to choose classes.

We know that $\mathscr{I}_Z|_{T_i}$ is T_i-flat for all i. So we conclude that $\mathscr{I}_Z$ is T_i-flat.

Lemma 1. *Assume $f : X' \to X$ is faithfully flat. A quasicoherent sheaf $\mathscr{F}$ on X is X-flat, respectively of finite presentation, etc, iff $f^*\mathscr{F}$ is.*

$h_{\mathscr{M}_5}(T) = \{W \subset \mathscr{O}_T \otimes V| \text{ with locally free cokernel}\}$. Again, the isomorphisms are unique if they exist, so the same descent argument applies.

So $h_{\mathscr{M}_3}, h_{\mathscr{M}_4}, h_{\mathscr{M}_5}$ are all sheaves.

$h_{\mathscr{M}_2} = \{\mathscr{L} \text{ on } X \times T\}/\simeq$, which is $\mathrm{Pic}(X \times T)$. The sheaf condition takes $\{T_i \to T\}$ and we want to check exactness for $\mathrm{Pic}(X \times Y) \to \prod \mathrm{Pic}(X \times T_i) \rightrightarrows \mathrm{Pic}(X \times T_i \times_T T_j)$.

But this is not exact at all!

Claim: Exactness ALWAYS fails on the left.

Proof. Choose T such that $\mathrm{Pic}(T) \neq \{0\}$. Let $\mathscr{M}$ be a nontrivial invertible sheaf on T. Then $p_2^*\mathscr{M} \in \mathrm{Pic}(X \times T)$ for $p_2 : X \times T \to T$.

Choose an open covering $T = \cup T_i$ such that $\mathscr{M}|_{T_i} \simeq \mathscr{O}_{T_i}$. So we take $\mathrm{Pic}(X \times T) \to \prod \mathrm{Pic}(X \times T_i)$ by $p_2^*\mathscr{M} \mapsto \mathscr{O}_{X\times T_i}$ and $\mathscr{O}_{X\times T} \mapsto \mathscr{O}_{X\times T_i}$ □

Claim: Exactness fails in the middle (in general)

Proof. $X/\mathbb{R}$ given by $x^2 + y^2 + z^2 = 0$ in $\mathbb{P}^2_{\mathbb{R}}$. We know that $X \otimes_{\mathbb{R}} \mathbb{C} \simeq \mathbb{P}^1_{\mathbb{C}}$ but $X \not\simeq \mathbb{P}^1_{\mathbb{R}}$. Thus there are no divisors of degree 1 (by Riemann-Roch).

Consider the covering $\operatorname{Spec}\mathbb{C} \to \operatorname{Spec}\mathbb{R}$. This gives $\mathrm{Pic}(X) \to \mathrm{Pic}(X \otimes \mathbb{C}) \rightrightarrows \mathrm{Pic}(X \otimes \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})$.

This is $\mathbb{Z} \xrightarrow{2} \mathbb{Z} \rightrightarrows \mathbb{Z} \times \mathbb{Z}$. so both of the maps to $\mathbb{Z} \times \mathbb{Z}$ take $1 \mapsto (1, 1)$, but 1 is not in th image. □

So Descent here fails because of the local nature of line bundles $\mathscr{L}$ on $X \times T'$, that $p_1^*\mathscr{L} \simeq p_2^*\mathscr{L}$ on $X \times T''$ and $\varphi_{jk} \circ \varphi_{ij} \neq \varphi_{ik}$, because there are too many isomorphisms.

We eventually realize we have one big problem: we are thinking of sets, rather than categories.

We fix this by shifting our thinking.

Definition 1 (Groupoid)**.** *A groupoid is a category where every morphism is invertible.*

Definition 2 (Discrete Groupoid)**.** *A groupoid $\mathcal{C}$ is discrete if $\forall x \in \mathcal{C}$, $\mathrm{Aut}(x) = \{\mathrm{id}\}$.*

Definition 3 (Connected Groupoid)**.** *A group is connected if any two objects are isomorphic.*

A group G is a groupoid with one object with $\mathrm{Aut}(x) = G$, this is an example of a connected groupoid.

A discrete groupoid is like a set, via $\chi : Sets \to Groupoids$ which takes a set S to a category with object set S and just identity maps.

Note: Groupoids do form a category, with arrows being functors.

Lemma 2. *The essential image of χ is the discrete groupoids.*

More good things: ${}_2(T)$ is a groupoid of $\mathscr{L}$ on $X \times T$. So $S \xrightarrow{f} T$ gives $\mathscr{M}_2(T) \xrightarrow{(\mathscr{L}_X f)^*} \mathscr{M}_2(S)$ functor takes $\mathscr{L}$ on $X \times T$ to $(\mathrm{id} \times f)^*\mathscr{L}$ on $X \times S$. We guess that this gives a functor $Sch^\circ \to Groupoids$

But composition gets funny, and is no longer on the nose.

$T'' \xrightarrow{g} T' \xrightarrow{f} T$ then there exists an isomorphism $g^*f^* \simeq (fg)^*$. This is the universal property of the pullback, which is unique up to unique isomorphism.

Exercise: What does all this MEAN?

Take $T''' \xrightarrow{h} T'' \xrightarrow{g} T' \xrightarrow{f} T$, and we get a commutative diagram of functors:

$$\begin{array}{ccc} & h^*(fg)^* & \\ \overset{\simeq}{\nearrow} & & \overset{\simeq}{\searrow} \\ h^*g^*f^* & & (fgh)^* \\ \overset{\simeq}{\searrow} & & \overset{\simeq}{\nearrow} \\ & (gh)^*f^* & \end{array}$$

Definition 4 (Fibered Category with Clivage (Pseudofunctor))**.** *A Fibered Category with Clivage, or Pseudofunctor, over a category $\mathcal{C}$ is*

1. *For each $c \in \mathcal{C}$, a groupoid $F(c)$*
2. *For each arrow $f : c \to d$ in $\mathcal{C}$, a functor $f^* : F(d) \to F(c)$*
3. *For each pair of arrows $c \xrightarrow{f} d \xrightarrow{g} e$, an isomorphism $\nu_{f,g} : f^*g^* \to (gf)^*$ such that the diagram above commutes with the isomorphisms being the ν's.*

<u>Olsson</u>

Let $A' \to A$ be a surjective map of rings with square zero kernel J, and $P' \to \mathrm{Spec}\, A'$ smooth scheme with reduction $P \to \mathrm{Spec}\, A$.

And $j : X \to P$ an inclusion with X smooth over $\mathrm{Spec}\, A$.

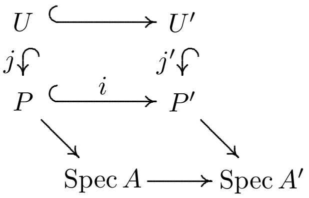
Problem: Understand how we can lift the picture $j : X \to P$ to a diagram $X' \hookrightarrow P'$ with X' smooth over $\mathrm{Spec}\, A'$.

Why? This would include lifting a variety with it's embedding into Projective Space.

We want $\mathscr{O}_{X'}$ on $|X|$ satisfying the following:

$$\begin{array}{ccccc} J \otimes \mathscr{O}_X & \dashrightarrow & \mathscr{O}_{X'} & \dashrightarrow & \mathscr{O}_X \\ & & \uparrow & & \uparrow \\ & & j^{-1}\mathscr{O}_{P'} & \longrightarrow & j^{-1}\mathscr{O}_P \\ & & \uparrow & & \uparrow \\ J & \longrightarrow & A' & \longrightarrow & A \end{array}$$

Define $\mathscr{L}$ to be a sheaf on $|X|$ which, to any open $U \subset X$, associates the set of diagrams

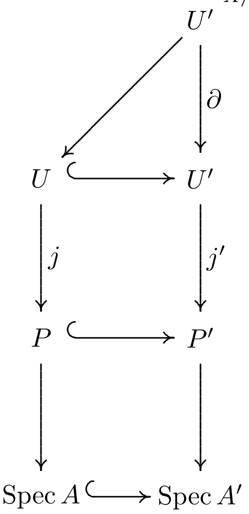


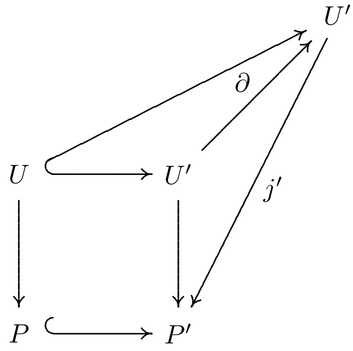
What are the global sections?

The set of arrows $U' \to P'$ filling in this diagram form a torsor under $\hom((i \circ j)^*\Omega^1_{P'/A'}, J \otimes \mathscr{O}_U) = \hom(j^*\Omega^1_{P/A}, J \otimes \mathscr{O}_U) = j^*T_{P/A} \otimes_A J$. There is an action of $j^*T_{P/A} \otimes J$ on $\mathscr{L}$.

If $I \subseteq \mathscr{O}_P$ is the ideal of X, then we look at the conormal bundle $j^*I = I/I^2 = \mathscr{N}^\vee$. We get an exact sequence $0 \to I/I^2 \xrightarrow{d} j^*\Omega^1_{P/A} \to \Omega^1_{X/A} \to 0$ and $0 \to T_{X/A} \to j^*T_{P/A} \to \mathscr{N} \to 0$. We tensor the second with J and we claim that $T_{X/A} \otimes J$ acts trivially on $\mathscr{L}$.

A section $\partial \in T_{X/A} \otimes J(U)$ corresponds to a diagram





Proposition 1. *$\mathscr{L}$ is a torsor under $\mathscr{N} \otimes J$.*

Torsor:

1. $\forall U \subset X$, there exists a covering $U = \cup U_i$ such that $\mathscr{L}(U_i) \neq \emptyset$
2. For all $U \subset X$ either $\mathscr{L}(U) = \emptyset$ or the action of $\mathscr{N} \otimes J(U)$ on $\mathscr{L}(U)$ is simply transitive.

Sketch of proof: Check that if U is affine, then the action of $\mathscr{N} \otimes J(U)$ on $\mathscr{L}(U)$ is simply transitive.

$0 \to T_{X/A} \otimes J(U) \to j^*T_{P/A} \otimes J(U) \to \mathscr{N} \otimes J(U) \to 0$.

General Fact: If G is a sheaf of abelian groups, then the set of isomorphism classes of G-torsors on $|X|$ are in bijection with $H^1(X, G)$.

In our case, we choose a covering of $X = \cup_i U_i$ with U_i affine and $s_i \in \mathscr{L}(U_i)$.

On $U_i \cap U_j$, we get two sections $s_i|_{U_ij}$ and $s_j|_{U_{ij}}$ in $\mathscr{L}(U_{ij})$.

The action of $\mathscr{N} \otimes J(U)$ on $\mathscr{L}(U_{ij})$ is simply transitive, and this implies that there exists a unique $x_{ij} \in \mathscr{N} \otimes J(U_{ij})$ such that $x_{ij} * s_i|_{U_{ij}} = s_j|_{U_{ij}}$. Now you check that $\{x_{ij}\}$ is a Cech 1-cocycle, so we get a class in $H^1(X, \mathscr{N} \otimes J)$.

So now $\mathscr{L}$ is trivial iff $\mathscr{L}(X) \neq \emptyset$ iff $[\mathscr{L}] \in H^1(X, \mathscr{N} \otimes J)$ is zero.

Summary:

1. There exists a canonical obstruction $o(j) \in H^1(X, \mathscr{N} \otimes J)$ whose vanishing is necessary and sufficient for the existence of a lifting of j.
2. The set of liftings j' of j form a torsor under $H^0(X, \mathscr{N} \otimes J)$ if $o(j) = 0$.

Reminder: $0 \to T_{X/A} \to j^*T_{P/A} \to \mathscr{N} \to 0$ induces

$$H^0(X, \mathscr{N} \otimes J) \to H^1(X, T_{X/A} \otimes J) \to H^1(X, j^*T_{P/A} \otimes J) \to H^1(X, \mathscr{N} \otimes J) \xrightarrow{\delta} H^2(X, T_{X/A} \otimes J)$$

. What is $\delta(o(j))$? It's $o(g)$ where g is the composition of j with the structure map $P \to \operatorname{Spec} A$.

Example: Let P be a smooth proper surface over k and $X \subset P$ a smooth rational curve with $X.X = -1$. By Hartshorne V.1.4.1, we get that $\deg \mathscr{N} = -1$. So $H^1(X, \mathscr{N} \otimes J) = 0$, $H^0(X, \mathscr{N} \otimes J) = 0$.

So $H^1 = 0$ says that there is no obstruction to X being lifted onto $P[\epsilon]$, and $H^0 = 0$ says that it is unique, and so it is $X[\epsilon]$.

Osserman

The Proof of Schlessinger's Theorem

(*) is $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$

H1 - (*) is surjective if $A'' \to A$ is a small thickening

H2 - (*) is bijective if $A'' = k[\epsilon]$ and $A = k$

H3 - T_F is finite dimensional

H4 - (*) is bijective if $A' = A''$ and $A' \to A$ is a small thickening

Theorem 1 (Schlessinger)**.** *Let F be a predeformation functor. Then F has a hull iff (H1),(H2),(H3) are satisfied.*

F is prorepresetable iff all four are satisfied.

Proposition 2. *Let F be a deformation functor and $A' \to A$ a small thickening with kernel I. For every $\eta \in F(A)$, when the set of $\eta' \in F(A')$ restricting to η is nonempty, it has a transitive action of $T_F \otimes_k I$. This action commutes with any morphism $F' \to F$ of deformation functors (H4) is satisfied iff for all $A' \to A$ small thickenings, and all $\eta \in F(A)$, this morphism is free (whenever the set is nonempty).*

Definition 5 (Essential)**.** *A surjection $p : A' \to A$ in $Art(\Lambda, k)$ is essential if $\forall q : A'' \to A$; such that pq is surjective, then q is surjective.*

Lemma 3. *If p is a small thickening, p is not essential iff p has a section.*

Example: $k[\epsilon] \to k$ is not essential, but $\mathbb{Z}/p^2 \to \mathbb{F}_p$ is.

Proposition 3. *If (H1)-(H3) are satisfied, then F has a hull*

Proof. We will proceed first by constructing a hull, then proving that it is one.

We'll construct (R, ξ) with $R \in \hat{Art}(\Lambda, k)$ and $\xi \in \hat{F}(R)$ such that $\bar{h}_R \xrightarrow{\xi} F$ is smooth and induces $T_R \simeq T_F$.

Let $\mathfrak{n}$ be the maximal ideal of Λ, $r = \dim T_F$ which is finite by (H3), then set $S = \Lambda[[t_1, \ldots, t_r]]$ and $\mathfrak{m}$ the maximal ideal of S.

We will construct R as S/J where $J = \cap_{i \geq 2} J_i$ and the J_i are constructed inductively.

$J_2 = \mathfrak{m}^2 + \mathfrak{n}S$, and $S/J_2 = k[T_S^*] = k[T_F^*] = k[\epsilon]^r$.

Set $R_2 = S/J_2$, and use (H2) to construct a $\xi_2 \in F(R_2)$ inducing a bijection $T_{R_2} \to T_F$.

So suppose we have $R_{i-1} = S/J_{i-1}$ and $\xi_{i-1} \in F(R_{i-1})$. We'll choose J_i to be minimal among J satisfying $\mathfrak{m}J_{i-1} \subseteq J \subseteq J_{i-1}$ and ξ_{i-1} can be lifted to an element of $F(R_i)$.

The first is preserved under arbitrary intersection, we need to check the second condition too.

Note: J satisfying the first condition corresponds to vector subspaces of $J_{i-1}/\mathfrak{m}J_{i-1}$, which is finite dimensional. This implies that it is enough to check finite (and thus pairwise) intersections.

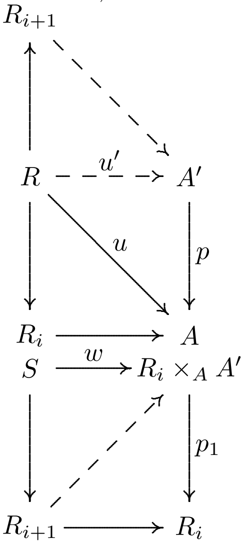
Suppose that J, K satisfy our conditions. Claim: $J \cap K$ does too. Again using $J_{i-1}/\mathfrak{m}J_{i-1}$, we can replace K without changing $J \cap K$ so that $J + K = J_{i-1}$. Then $S/J \times_{S/J_{i-1}} S/K = S/(J \cap K)$. So by (H1), we have some element of $F(S/(J \cap K))$ restricting to ξ_{i-1}, which means $J \cap K$ satisfies our conditions.

So we can set J_i to be the minimal ideal satisfying the conditions. As stated before, we define $J = \cap_{i \geq 2} J_i$ and $R = S/J$.

If $R_i = S/J_i$, because $\mathfrak{m}^i \subseteq J_i$, we have $R = \varprojlim R/J_i$ and furthermore, there is an element ξ which is the limit of the ξ_i.

So (R, ξ) is our prospective hull. $T_R \simeq T_F$ is immediate from the choice of ξ_2, smoothness is harder. Fix $p : A' \to A$ a small thickening, $\eta' \in F(A')$ such that $p(\eta') = \eta \in F(A)$ and $u : R \to A$ such that $u(\xi) = \eta$. Want a lift $u' : R \to A'$ such that $u'(\xi) = \eta'$

First, construct any u' lifting u. Since A is an Artin ring, it factors through $R \to R_i$ for some i.



with p_1 a small thickening. If we have a section, then no problem. If not, then $p_!$ is essential, so we choose w as above, must be surjective. Enough to show $\ker w \supset J_{i+1}$. This follows from (H1).

So we have some u', we want to have $u'(\xi) = \eta'$. But we have compatible transitive actions of $T_F \otimes I \simeq T_R \otimes I$ of $F(p)^{-1}(\eta)$ and $h_R(p)^{-1}(\eta) = R \to A'$ such that $R \to A$ sends ξ to η.

So $\exists \tau \in T_F \otimes I$ sending $u'(\xi)$ to η'. Then we can modify u' by τ and we'll have the desired u' lifting u and sending ξ to η. $\square$