Lieblich

Descent Theory = Gluing

Well, in the Zariski topology, if we have sheaves defined on open sets, we can glue them together to get a sheaf on the whole thing.

So at the Zariski level if X is a scheme and $\{U \subset X\}$ is an open covering with $\mathscr{F}_i$ on U_i quasi-coherent sheaves, we need maps $\varphi_{ij} : \mathscr{F}_i|_{U_i \cap U_j} \to \mathscr{F}_j|_{U_i \cap U_j}$ is an isomorphism such that $\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$ for all i, j, k on $U_i \cap U_j \cap U_k$.

Definition 1 (Descent Datum). *Let $f : X' \to X$ be an fpqc (faithfully flat, quasi-compact).*
$X'' = X' \times_X X'$

$$\begin{array}{ccc} & X'' & \\ {}^{p_1}\swarrow & & \searrow^{p_2} \\ X' & & X' \end{array}$$

And $\mathscr{F}$ a quasi-coherent sheaf on X'. Then a descent datum is an isomorphism $\varphi : p_1^*\mathscr{F}' \to p_2^*\mathscr{F}'$ such that $p_{23}^*\varphi \circ p_{12}^*\varphi = p_{13}^*\varphi$.

$$\begin{array}{ccccc} p_{12}^*p_1^*\mathscr{F}' & \longrightarrow & p_{12}^*p_2^*\mathscr{F}' & & \\ & & \downarrow \simeq & & \\ \downarrow \simeq & & p_{23}^*p_1^*\mathscr{F}' & \longrightarrow & p_{23}^*p_2^*\mathscr{F}' \\ & & & & \downarrow \simeq \\ p_{13}^*p_1^*\mathscr{F} & & \longrightarrow & & p_{13}^*p_2^*\mathscr{F}' \end{array}$$

Reinterpretation: A descent datum on $\mathscr{F}'$ consists of an isomorphism $\varphi_{t_1,t_2} : t_1^*\mathscr{F}' \simeq t_2^*\mathscr{F}'$ for all $t_1, t_2 \in X'(T)$ for fixed $T \in Sch_X$, such that $\forall t_1, t_2, t_3 \in X'(T)$, $\varphi_{t_2,t_3} \circ \varphi_{t_1,t_2} = \varphi_{t_1,t_3}$ and this is functorial in T, t_i. (this includes that $\varphi_{t,t} = \mathrm{id}$)

Note: If $\mathscr{F}' = f^*\mathscr{F}$ then there is a natural descent datum $\varphi_{t_1,t_2} : t_1^* f^*\mathscr{F} \simeq t_2^* f^*\mathscr{F}$ and $ft_1 = ft_2$ and $(ft_1)^* \simeq t_1^* f^*$ naturally, so we get natural descent data as above. We denote this by $(f^*\mathscr{F}, can)$.

Definition 2 (Category of Descent Data for f). *The category of Descent data for f, $\mathcal{D}_f$ is the category of pairs $(\mathscr{F}', \varphi)$ where $\mathscr{F}'$ is a quasi-coherent sheaf on X' and φ is a descent datum.*

The maps are $\psi : \mathscr{F}_1' \to \mathscr{F}_2'$ such that

$$\begin{array}{ccc} p_1^*\mathscr{F}_1' & \xrightarrow{\phi_1} & p_2^*\mathscr{F}_2' \\ \downarrow p_1^*\psi & & \downarrow p_2^*\psi \\ p_1^*\mathscr{F}_2' & \xrightarrow{\phi_2} & p_2^*\mathscr{F}_2' \end{array}$$

Note then that pullback defines a functor $\tilde{f}^* : QCoh(X) \to \mathcal{D}_f$ by $\mathscr{F} \mapsto (f^*\mathscr{F}, can)$. This functor will tell us how gluable things are.

Definition 3 (Descent Morphism). *f is a descent morphism if $\tilde{f}^*$ is fully faithful.*

f is an effective descent morphism if $\tilde{f}^*$ is an equivalence.

Theorem 1 (Grothendieck). *If $f : X' \to X$ is fpqc, then f is an effective descent morphism for quasi-coherent sheaves.*

That is, we are allowed to glue using these morphisms.

Theorem 2 (Girard/Grothendieck)**.** *If f has a section then f is an effective descent morphism.*

Proof. Let $f : X' \to X$ be a morphism, $\sigma : X \to X'$ a section. We will show that $\tilde{f}^*$ is fully faithful and essentially surjective.

$\tilde{f}^*$ is clearly faithful, as $\sigma^* f^* = \mathrm{id}$, $\alpha : \mathscr{F} \to \mathscr{G}$ such that $f^*\alpha = 0$ implies that $\sigma^* f^* \alpha = \alpha = 0$.

$\tilde{f}^*$ is full: a map $(\mathscr{F}, \varphi) \xrightarrow{\psi} (\mathscr{F}', \varphi')$. That is, we have maps $t : T \to X'$ and $T \to X$ and $t^*\mathscr{F} \xrightarrow{\psi_t} t^*\mathscr{F}'$ such that for all t_1, t_2 we get the square above.

Ref. point=σ_T, with σ the section. So we get the following diagram:

$$\begin{array}{ccc} \sigma^*\mathscr{F} & \xrightarrow{\psi_\sigma = \sigma^*\psi} & \sigma^*\mathscr{F}' \\ \downarrow{\scriptstyle \varphi_{\sigma,t}} & & \downarrow{\scriptstyle \varphi'_{\sigma,t}} \\ t^*\mathscr{F} & \xrightarrow{\psi_t} & t^*\mathscr{F}' \end{array}$$

So we can propagate ψ_σ.

Now all that remains is to check essentially surjective: Fix $(\mathscr{F}, \varphi) \in \mathcal{D}_f$ and $t \in X'(T)$, $T \in Sch_X$.

We hope that $(\mathscr{F}, \varphi) \simeq \tilde{f}^*(\sigma^*\mathscr{F}) = (f^*\sigma^*\mathscr{F}, can)$

$\varphi_{t,\sigma f t} : t^*\mathscr{F} \to t^* f^* \sigma^* \mathscr{F}$. Given t_1, t_2, we get the diagram

$$\begin{array}{ccc} t_1^*\mathscr{F} & \xrightarrow{\varphi_{t_1,\sigma f t_2}} & t_1^* f^* \sigma^* \mathscr{F} \\ \downarrow{\scriptstyle \varphi_{t_1,t_2}} & & \downarrow{\scriptstyle \varphi_{\sigma f t_1, \sigma f t_2} = can} \\ t_2^*\mathscr{F} & \xrightarrow{\varphi_{t_2,\sigma f t_2}} & t_2^* f^* \sigma^* \mathscr{F} \end{array}$$

This comes from gluing for $\mathscr{F}$. And as when we showed that this was full, we have an isomorphism $\mathscr{F}$ to $f^*\sigma^*\mathscr{F}$ respecting the canonical descent datum. □

We now prove the previous theorem:

Proof. Special case: X, X' affine, $\operatorname{Spec} B \to \operatorname{Spec} A$, $A \to B$ faithfully flat.

1. $\tilde{f}^*$ is fully faithful iff for M, N being A-modules, we have that $\hom_A(M, N) \to \hom_B(M \otimes_A B, N \otimes_A B) = \hom_A(M, N \otimes B) \rightrightarrows \hom_{B \otimes_A B}(M \otimes_A B \otimes_A B, N \otimes_A B \otimes_A B) = \hom_A(M, N \otimes_A B \otimes_A B)$ is exact, so we get $\hom_A(M, N \to N \otimes_A B \rightrightarrows N \otimes_A B \otimes_A B)$, so just need to show that $N \to N \otimes_A B \rightrightarrows N \otimes_A B \otimes_A B$ is exact. We reduce to the case where $B \to A$ is a section, and then it is fairly straightforward.

2. $\tilde{f}^*$ is essentially surjective: $(\mathscr{F}, \varphi)$ is M and $\varphi : B \otimes_A M \to M \otimes_A B$ an isomorphism of $B \otimes_A B$-modules Guess what $\mathscr{F}$ on X should be such that $\tilde{f}^*(\mathscr{G}) \simeq (\mathscr{F}, \varphi)$. We take $N = \{m \in M | m \otimes 1 = \varphi(1 \otimes m)\}$. Observe that there is a map $\nu : N \otimes_A B \to M$ which is compatible with the descent data.

   So we may assume that there is a section $X \to X'$ (from $B \to A$ an augmentation). Now we know that descent is effective, and so the proof in this case shows that ν is an isomorphism in $\mathcal{D}_f$.

□

<u>Olsson</u>

<u>Obstruction Theories</u>

Let $\pi : A' \to A$ be a surjection of rings and $I = \ker \pi$ a square zero ideal as an A-module.

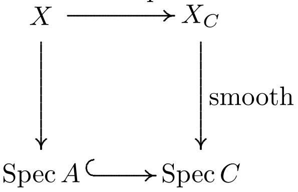
Let $g : X \to \operatorname{Spec} A$ be a smooth separated scheme.

Problem, we want to understand the liftings

$$\begin{array}{ccc} X & \longrightarrow & X' \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g'} \\ \operatorname{Spec} A & \hookrightarrow & \operatorname{Spec} A' \end{array}$$

with g' smooth.

We defined $\mathrm{Def}_X : Alg/A \to Set$ yesterday by $(f : C \to A) \mapsto$ the set of isomorphism classes of squares over this map:



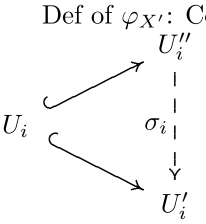
Yesterday, we said that $T_{\mathrm{Def}_X} = H^1(X, T_{X/A})$.

When X is affine, we have the following:

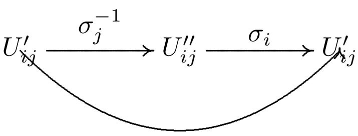
1. $\exists$ a lifting $X' \to \mathrm{Spec}\, A'$
2. Any two liftings are isomorphic
3. the group of automorphism of any lifting $X' \to \mathrm{Spec}\, A'$ is canonically isomorphic to $H^0(X, T_X \otimes I)$

For general X, if $X' \to \mathrm{Spec}\, A'$ is a smooth lifting, I get a bijection $varphi_{X'} : \mathrm{Def}_X(A' \to A) \to H^1(X, T_X \otimes I)$. Yesterday the fixed lifting was $X[I] \to \mathrm{Spec}\, A[I]$. This lifting cannot be chosen canonically, however.

Def of $\varphi_{X'}$: Cover X by U_i affine sets, and take $X'' \in \mathrm{Def}_X(A')$. Then for all i we get



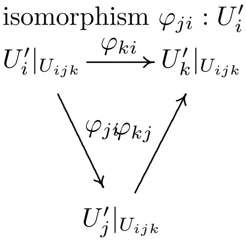
Choosing $\sigma_i : U''_i \to U'_i$ for each i gives, for all i, j,



??????

Q: When $\exists X' \to \mathrm{Spec}\, A'$?

Let $\mathcal{U} = \{U_i\}$ be a covering of X by affines. Fix listing maps $U'_i \to \mathrm{Spec}\, A'$. For all i, j, choose an isomorphism $\varphi_{ji} : U'_i|_{U_{ij}} \to U'_j|_{U_{ij}}$



This diagram need not commute! We in fact get $\partial_{ijk} = \varphi_{ki}^{-1} \circ (\varphi_{kj} \circ \varphi ji) \in H^0(U_{ijk}, T_X \otimes I)$, an automorphism.

Lemma 1. *1. $\{\partial_{ijk}\}$ is a Cech 2-cocycle*

2. If φ'_{ji} is a second choice of isomorphisms that give $\{\partial'_{ijk}\}$, then $\{\partial_{ijk}\} - \{\partial'_{ijk}\}$ is a Cech boundary, $o(g) \in H^2(X, T_X \otimes I)$.

Proposition 1. *∃ a lifting* $X' \xrightarrow{g'} \operatorname{Spec} A'$ *of* g *iff* $o(g) = 0$.

Summary:

1. There is a canonical obstruction $o(g) \in H^2(X, T_X \otimes I)$ such that $o(g) = 0$ iff $\operatorname{Def}_X(A' \to A) \neq \emptyset$
2. If $o(g) = 0$, then the set of isomorphism classes of liftings form a torsor under $H^1(X, T_X \otimes I)$
3. For any lifting of g, the group of automorphisms is canonically isomorphic to $H^0(X, T_X \otimes I)$.

Take Λ a ring, $F : \Lambda - alg \to Set$.

Definition 4 (Obstruction Theory)**.** *An obstruction theory for* F *consists of the following data:*

1. *∀ morphisms* $A \to A_0$ *of* Λ*-algebras with kernel a nilpotent ideal and* A_0 *reduced and* $a \in F(A)$*, we get a functor* $\mathscr{O}_a$: *(finite type* A_0*-modules)* $\to$ *(finite type* A_0*-modules).*
2. *For all diagrams* $A' \to A \to A_0$ *and* $a \in F(A)$*, where* $A' \to A$ *is surjective with* $\ker(A' \to A) = J$ *annihilated by* $\ker(A' \to A_0)$*, a class* $o(a) \in \mathscr{O}_a(J)$ *which is zero iff* a *lifts to* $F(A')$*.*

   Generally A_0 *is a field,* A *is a thickening, and* A' *is a further thickening.*

   This should be functorial in the natural way.

Example: Let $j : X \hookrightarrow X'$ be a closed immersion defined by a square zero ideal J, L a line bundle on X.
Problem: Understand the liftings of L to L' on X.

We could take a lifting to a line bundle isomorphic to L, or we could take a pair (L', ι) where L' is a line bundle on X' and $\iota : j^*L' \to L$ is a specific isomorphism of line bundles on X.

We will use the latter, and two are isomorphic $(L', \iota) \simeq (L'', \epsilon)$ if there exists $\sigma : L' \to L''$ an isomorphism such that $\epsilon\sigma = \iota$ (abusing notation to say σ is a map $j^*L' \to j^*L''$.) Note that just because $(L', \iota), (L', \epsilon)$ exist, there doesn't NEED to be an homomorphism between them because the diagram may not commute.

No we have $0 \to J \to \mathscr{O}^*_{X'} \to \mathscr{O}^*_X \to 0$ with the first map $g \mapsto 1 + g$, is a sequence of sheaves on $|X|$, and $(1 + f)(1 + g) = 1 + (g + f) + gf = 1 + (g + f)$.

This sequence is exact, so we get $0 \to H^0(J) \to H^0(\mathscr{O}^*_{X'}) \to H^0(\mathscr{O}^*_X) \to H^1(J) \to \operatorname{Pic}(X') \to \operatorname{Pic}(X) \to H^2(J) \to \dots$ from the long exact sequence on cohomology.

Proposition 2. *Assume* $H^0(X', \mathscr{O}^*_{X'}) \to H^0(X, \mathscr{O}^*_X)$ *is surjective. (Then the map* $H^0(\mathscr{O}^*_X) \to H^1(J)$ *is zero)*

1. *There exists an obstruction* $o(L) = \partial[L] \in H^2(X, J)$ *which is* 0 *iff there exists* (L', ι)*.*
2. *If* $o(L) = 0$*, then the set of isomorphism classes of liftings* (L', ι) *is a torsor under* $H^1(X, J)$
3. *For all liftings, the group of automorphisms is in canonical bijection with* $H^0(X, J)$*.*

Osserman
Remarks: Fiber products may seem strange. We'll come back to it later.
(H1) and (H2) of Schlessinger's criterion are almost always satisfied if you've written down the deformation functor properly.
(H3) tends to be related to some sort of properness hypothesis.
(H4) is related to presence of automorphisms.

Definition 5 (Deformation Functor)**.** *A predeformation functor* F *is a deformation functor if it satisfies (H1) and (H2).*

Note: (H3) Makes sense for any Deformation functor.

Definition 6 (Automorphism of Pair)**.** *Given* $(X_A, \varphi) \in \operatorname{Def}_X(A)$*, an automorphism of* (X_A, φ) *(an infinitesimal automorphism of* X_A*) is an automorphism of* X_A *over* A *commuting with* φ*.*

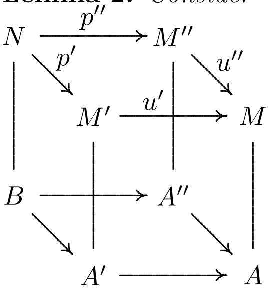
Theorem 3. *Let X be a scheme over k and Def_X the functor of deformations of X. Then:*

1. Def_X *is a deformation functor*
2. Def_X *satisfies (H3) if X is proper.*
3. Def_X *satisfies (H4) iff for all $A' \to A$ small thickening, the pair $(X_{A'}, \phi)$ over A', every automorphism of $(X_{A'}|_A, \phi|_A)$ is the restriction of an automorphism of (X_A, ϕ).*
   In particular, if $H^0(X, \mathscr{H}om(\Omega^1_{X/k}, \mathscr{O}_X)) = 0$, then (H4) is satisfied.

Corollary 1. *If X is proper, then Def_X has a hull, and if further $H^0(X, \mathscr{H}om(\Omega^1_{X/k}, \mathscr{O}_X)) = 0$, then Def_X is prorepresentable.*

Example: If X is a smooth, proper curve, then Def_X has a hull, and Def_X is prorepresentable if $g \geq 2$.

Lemma 2. *Consider*



of compatible ring and module homomorphisms, and with $B = A' \times_A A''$, $N = M' \times_M M''$ and M' and M'' are flat over A' and A'' and $A'' \to A$ surjective with nilpotent kernel and u' induces $M' \otimes_{A'} A \simeq M$ and similarly for u''.

Then, N is flat over B and p' induces $N \otimes_B A' \simeq M$ and similarly for p''. Also, in same situation, if we have L a B-module and $q : L \to M'$ and $q'' : L \to M''$ such that q' induces $L \otimes_B A' \simeq M'$, then $q' \times q'' : L \to N$ is an isomorphism.

Note that this is more general than is necessary for Schlessinger, since we don't restrict to Artin local rings. Here, the modules are all free. The general argument uses the local criterion for flatness.

Proposition 3. *Given $A' \to A$, $A'' \to A$ where $A'' \to A$ is surjective with nilpotent kernel, write $B = A' \times_A A''$, then*

1. *Given X' and X'' flat over A' and A'', and an isomorphism $\varphi : X'|_A \to X''|_A$ there exists Y flat over B with maps $\phi' : X' \to Y$ and $\phi'' : X'' \to Y$ (over the map $\mathrm{Spec}\, A' \to \mathrm{Spec}\, B$) inducing isomorphisms $X' \to Y|_{A'}$ and $X'' \to Y|_{A''}$ and $\phi = \phi''|_A \circ {\phi'}^{-1}|_A$.*
2. *Given Y_1, Y_2 flat over B, the natural map $Isom_B(Y_1, Y_2) \to Isom_{A'}(Y_1|_{A'}, Y_2|_{A'}) \times_{Isom_A(Y_1|_A, Y_2|_A)} Isom_{A''}(Y_1|_{A''}, Y_2|_{A''})$ is a bijection.*

Proof.
1. We'll consider Y on the same topological space as X'. We identify the spaces of X'' and $X''|_A$ and also $X'|_A$ using ϕ and write $i : X|_A \to X'$. Set $\mathscr{O}_Y(U) = \mathscr{O}_{X'}(U) \times_{\mathscr{O}_{X'|_A}(i^{-1}(U))} \mathscr{O}_{X''}(i^{-1}U)$. So $\mathscr{O}_Y = \mathscr{O}_{X'} \times_{i_* \mathscr{O}_{X'|_A}} \mathscr{O}_{X''}$.

   The lemma tells us that $\mathscr{O}_Y$ is flat over B and that it recovers $\mathscr{O}_{X'}$ and $\mathscr{O}_{X''}$ upon restriction to A' and A''. The recovery of ϕ is a diagram chase. We also check that $\mathscr{O}_Y$ is a sheaf and defines a scheme structure, which boils down to module fiber product commutes with localization.
2. Similar, but use second part of the lemma.

□

We now prove theorem about Def_X.

Proof.

1. (H1) follows from part 1 of the proposition. (H2) uses part 2 of prop, also uses $A = k$ so the φ in definition of Def_X rigidify the isomorphisms
2. Is true for smooth X from Martin's lecture, see later lectures for general statements.
3. Is similar to 1.

□