Lieblich

Let X be a scheme. Then h_X has a nice property. Fix Y, and take $U \subset Y$ to $\hom(U, X) = h_X(U)$. This is a sheaf in the Zariski Topology.

$\{U_i \subset Y\}$ is an open cover, then we have $h_X(Y) \xrightarrow{a} \prod_i h_X(U_i) \overset{\xrightarrow{b}}{\underset{c}{\rightarrow}} \prod_{i,j} h_X(U_i \cap U_j)$ is exact. That is, a is injective and $\mathrm{Im}(a) = \{\alpha | b(\alpha) = c(\alpha)\}$.

Problem: Zariski Topology is not "geometric."

Serre said in FAC that it was good for Sheaves. However, Grothendieck wanted an abstract categorical notion of topology.

Observation: If X is a topological space, get a category with objects $U \subseteq X$ open with arrows $\hom(U, V)$ either the empty set or containing one arrow if $U \subset V$.

So then what is a presheaf? Just a contravariant functor from this category to Sets.

For sheaves, we need a bit more, we need to know what an open covering is, and how to glue things. We want to retain $\{V_i \subseteq U\}$ an open cover, that is, a set of arrows $V_i \to U$ in the category.

Silly Properties of Coverings

1. $\{U \subseteq U\}$ is a covering
2. If $\{V_i \subset U\}$ is a covering and $W \subseteq U$, then $(V_i \cap W) \subseteq W$ is a covering.
3. If $\{V_i \subset U\}$ is a covering and $\{W_{ij} \subset V_i\}$ are coverings then $\{W_{ij} \subset U\}$ is a covering.

Definition 1 (Grothendieck Topology)**.** *Given a category $\mathcal{C}$, a Grothendieck Topology is a collection of sets of arrows $\{V_i \to U\}$ for each $U \in \mathcal{C}$, called coverings, such that*

1. *Any isomorphism is a covering*
2. *If $\{V_i \to U\}$ is a covering and $W \to U$, then $V_i \times_U W$ exists for each i and $\{V_i \times_U W \to W\}$ is a covering*
3. *If $\{W_{ij} \to V_i\}$ is a covering and $\{V_i \to U\}$ is a covering, then $\{W_{ij} \to U\}$ is a covering.*

A category with a Grothendieck Topology is called a Site.

Examples: X is a scheme. X_{Zar} is the small Zariski Topology. The objects are open immersions into X and the arrows are maps which commute with the immersion maps, and the coverings are collections of arrows $\phi_i : V_i \to U\}$ such that $\bigcup \phi_i(V_i) = U\}$.

X_{ZAR} is the big Zariski site. $\mathcal{C} = Sch_X$. Let $Z \in \mathcal{C}$. Then the coverings are collections $\{\phi_i : Y_i \to Z\}$ of arrows of X-morphisms such that each ϕ_i is an open immersion and $\bigcup \phi_i(Y_i) = Z$.

$X_{\acute{e}t}$ the small étale site. $\mathcal{C} = \{Z \to X \text{ étale}\}$ is a subcat of Sch_X. Then coverings are $\phi_i : Y_i \to Z$ X-morphisms with $\bigcup \phi_i(Y_i) = Z$.

$X_{\acute{E}T}$ the big étale site where $\mathcal{C} = Sch_X$ and the coverings of Z are $\phi_i : Y_i \to Z$ with ϕ_i étale and $\bigcup \phi_i(Y_i) = Z$.

X_{fppf} is the fppf site. $\mathcal{C} = Sch_X$ and the coverings are $\phi_i : Y_i \to Z$ with ϕ_i flat and locally of finite presentation with $\bigcup \phi_i(Y_i) = Z$.

Definition 2 (Sheaf of Sets)**.** *Given a site $\mathcal{C}$, a sheaf of sets on $\mathcal{C}$ is a functor $F : \mathcal{C}^\circ \to Set$ such that for all coverings $\{Y_i \to Z\}$ in $\mathcal{C}$ the diagram*

$$F(Z) \to \prod_i F(Y_i) \rightrightarrows \prod_{i,j} F(Y_i \times_Z Y_j)$$

is exact.

We do need to consider $i = j$, eg $\operatorname{Spec} \mathbb{Q}(\sqrt{2}) \to \operatorname{Spec} \mathbb{Q}$.

Theorem 1 (Grothendieck)**.** *For any X-scheme S, the functor $h_X : Sch_X^\circ \to Set$ is an fppf sheaf.*

Proof. Fix $\{Y_i \to Z\}$ a covering, and show $h_S(Z) \to \prod h_S(Y_i) \rightrightarrows h_S(Y_i \times_Z Y_j)$ is exact.

Baby case; $\{Y_i \to Z\}$ is $\operatorname{Spec} B \to \operatorname{Spec} A$, $A \to B$ is faithfully flat and $S = \operatorname{Spec} C$.

Diagram becomes $\hom(C, A) \to \hom(C, B) \rightrightarrows \hom(C, B \otimes_A B)$. $\hom(C, A \to B \rightrightarrows B \otimes_A B)$.

Lemma 1. *$A \to B \rightrightarrows B \otimes_A B$ is exact (top map is $b \mapsto b \otimes 1$ and bottom is $b \mapsto 1 \otimes b$).*

The lemma is equivalent to $0 \to A \to B \to B \otimes_A B$ exact, where the map is $b \mapsto b \otimes 1 - 1 \otimes b$. We look at the special case where there exists $\sigma : B \to A$ such that $A \to B \xrightarrow{\sigma} A$ is the identity. We the get $B \otimes_A B \to B$ by $b \otimes C \mapsto \sigma(b)c$. We show that if $b \otimes 1 = 1 \otimes b$ then $b \in A$ this condition implies that $\sigma(b) = b$, so we get a $\sigma(b) \in A$ is equal to b, so $b \in A$.

Observe that to prove $0 \to A \to B \to B \otimes_A B$ is exact, it is enough to prove it after a faithfully flat base change $A \to D$. Let $D = B$, then $A \to B$ becomes $B \to B \otimes_A B \to B$, which reduces the problem to the special case. [Check This].

So the lemma is established.

We will use the following lemma without proof:

Lemma 2. *$F : Sch_X^\circ \to Set$ is an fppf sheaf iff it is a Zariski Sheaf and for all $SpecB \to SpecA$ with $A \to B$ faithfully flat and of finite presentation, then $F(V) \to F(U) \rightrightarrows F(U \times_V U)$.*

Corollary 1. *If S if affine, then h_S is an fppf sheaf.*

Sketch of the general case: Let $S_i \subseteq S$ be an affine covering of S. Let $U \to V$ be an fppf covering. Then we want to look at $h_S(V) \to h_S(U) \rightrightarrows h_S(U \times_V U)$. $U \to S$ such that the two maps $U \times_V U \to U \to S$ must agree. We then get, using fppf, that there exists $|V| \to |S|$ such that $|U| \to |V| \to |S|$ corresponds to our original $U \to S$.

We pull back $S_i \subset S$ and take $V_i = f^{-1}(S_i)$ and $U_i = U \times_V V_i$. Then $U_i \times_{V_i} U_i \rightrightarrows U_i \to V_i$ and $U_i \to S_i$ affine causes a map $V_i \to S_i$ to exist, and these glue to $V \to S$ as desired. □

<u>Olsson</u>

$A \to R$ a ring homomorphism, take the category $A - alg/R$ of diagrams

$$\begin{array}{ccc} C & \xrightarrow{f} & R \\ \uparrow & \nearrow & \\ A & & \end{array}$$

And take $F : A - alg/R \to Set$ a functor. If For all $I, J \in Mod_R$ the natural map $F(R[I \oplus J]) \to F(R[I]) \times F(R[J])$ is an isomorphism, then we get a tangent space T_F.

[In fact, $\forall I, F(R[I])$ is an R-module and $T_F := F(R[\epsilon])$]

$+ : F(R[\epsilon]) \times F(R[\epsilon]) \simeq F(R[\epsilon_1, \epsilon_2]/(\epsilon_1^2, \epsilon_2^2, \epsilon_1\epsilon_2)) \to F(R[\epsilon])$ with the map by $\epsilon_i \to \epsilon$.

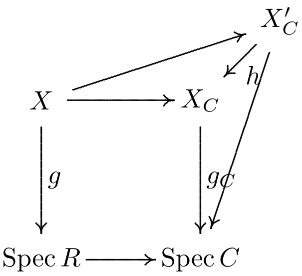
$\times f : F(R[\epsilon]) \to F(R[\epsilon])$ is induced by $R[\epsilon] \to R[\epsilon]$ by $a + b\epsilon \mapsto a + fb\epsilon$.

Problem 1: R a ring, $g : X \to \operatorname{Spec} R$ a separated, smooth map. Consider the functor $\operatorname{Def}_X : \mathbb{Z}{-}Alg/R \to Set$ with $\operatorname{Def}_X(C \xrightarrow{f} R) =$ the set of isomorphism classes of cartesian diagrams

$$\begin{array}{ccc} X & \longrightarrow & X_C \\ \downarrow g & & \downarrow g_C \\ \operatorname{Spec} R & \longrightarrow & \operatorname{Spec} C \end{array}$$

with g_C smooth.

A morphism of diagrams is an arrow $h : X'_C \to X_C$ such that the following commutes



Call the above Diagram 1.

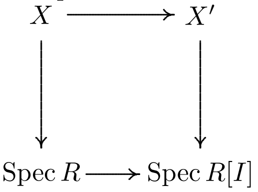
Remark: If $C = R[I]$ for some R-module I, then any morphism h as in Diagram 1 is an isomorphism.

Proposition 1. *For all* $I, J \in Mod_R$, $\operatorname{Def}_X(R[I \oplus J]) \to \operatorname{Def}_X(R[I]) \times \operatorname{Def}_X(R[J])$ *is an isomorphism.*

Proof in Osserman on Day 3.

How to compute T_{Def_X} or more generally the R-module $\operatorname{Def}_X(R[I])$?

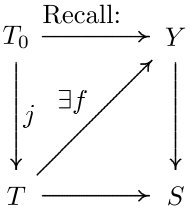
Special Case: X affine. Facts: (1) $\operatorname{Def}_X(R[I])$ consists of one element and (2) For any deformation



then the set of maps $h : X' \to X'$ as in Diagram 1 is in canonical bijection with $H^0(X, T_X \otimes I)$.

Why is there a lifting? $X = \operatorname{Spec} R[x_1, \dots, x_r]/(f_1, \dots, f_\ell)$. In fact, $X[I] \to \operatorname{Spec} R[I]$ is a smooth lifting. REASON???

Recall:



j a closed immersion defined by a square-zero ideal J then the set of arrows f filling in the diagram is a pseudo-torsor under $\hom(f_0^*\Omega^1_{Y/S}, J)$

Definition 3 (Pseudo-Torsor)**.** *Either no arrow exists or, if an arrow exists, there is a simply transitive action of* $\hom(f_0^*\Omega^1_{Y/S}, J)$ *on the set of arrows.*

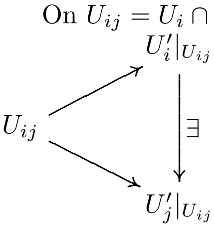
For general $X \to \operatorname{Spec} R$, this also shows that $(X[I] \to \operatorname{Spec} R[I]) \in \operatorname{Def}_X(R[I])$. Choose a covering $X = \bigcup_i U_i$ with each U_i affine and denote $\{U_i\} = \mathcal{U}$. Choose for each $i \in I$ a smooth lifting $U'_i \to \operatorname{Spec} R[I]$

We want to patch the U'_i to a lifting of X.

$X \to X'$ corresponds to $\mathscr{O}_{X'} \overset{I \otimes_R \mathscr{O}_X}{\to} \mathscr{O}_X$ on $|X|$

$U'_i \to \operatorname{Spec} R[I]$ corresponds to $\mathscr{O}_{U'_i} \overset{I \otimes_R \mathscr{O}_{U_i}}{\to} \mathscr{O}_{U_i}$ on $|U_i|$.

On $U_{ij} = U_i \cap U_j$, we get a diagram



two elements of $\operatorname{Def}_{U_{ij}}(R[I])$.

Pick $\forall i$ and isomorphism $\sigma_i : U'_i \to U_i[I]$. Note that any other choice of σ_i is given by composition with an automorphisms of $U_i[I] \leftrightarrow H^0(U_i, T_X \otimes I)$.

There is an obstruction for the σ_i's to glue to an isomorphism $X' \simeq X[I]$. $x_{ij} : U_{ij}[I] \overset{\sigma_j^{-1}}{\to} U'_{ij} \overset{\sigma_i}{\to} U_{ij}(U)$.

Lemma 3. *$x_{ij} = x_{ij} + x_{jk}$ in $H^0(U_{ijk}, T_X \otimes I)$.*

Proof. Simple diagram chase. □

Corollary 2. *The $\{x_{ij}\}$ define a Cech Cocycle $[X'] \in \check{H}^1(X, T_X \otimes I) = H^1(X, T_X \otimes I)$.*

Theorem 2. *The map $\mathrm{Def}_X(R[I]) \to H^1(X, T_X \otimes I)$ by $X' \mapsto [X']$ is an R-module isomorphism.*

Osserman

Examples: For "nice" global moduli functors, it works well to simply restrict to $Art(\Lambda, k)$ to obtain predeformation functors. An example is deformations of a closed subscheme.

Let X_Λ be a scheme over $\operatorname{Spec}\Lambda$ and write X for $X_\Lambda|_{\operatorname{Spec} k}$. Let $Z \subseteq X$ be a closed subscheme. $\mathrm{Def}_{Z,X} : Art(\Lambda, k) \to Set$ is defined by $A \mapsto \{Z_A \subset X_\Lambda|_{\operatorname{Spec} A}$ closed subscheme, flat over A, such that $Z_A|_{\operatorname{Spec} k} = Z\}$.

Sometimes, simple restriction of functors isn't so good.

Example: Deformations of a scheme. Fix X over k, then Def_X is defined by $A \mapsto \{(X_A, \varphi) : X_A$ is flat over $\operatorname{Spec} A$, $\varphi : X \to X_A$ such that φ induces isomorphism of X with the fiber product $X_A \times_A k\}/\simeq$.

Note: If we naively restrict functors, we still get a predeformation functor, but its behavior will be worse.

Problem comes from the automorphisms of X not extending to X_A. This is the first hint that for moduli problems involving automorphisms that functors to Sets don't capture everything.

Example: Deformation of a quasicoherent sheaf. Fix X_Λ over $\operatorname{Spec}\Lambda$ and set $X = X_\Lambda|_k$ fix $\mathscr{E}$ a quasicoherent sheaf on X. Define $\mathrm{Def}_{\mathscr{E}}$ by $A \mapsto \{(\mathscr{E}_A, \phi)|\mathscr{E}_A\}$ is quasicoherent on $X_\Lambda|_A$, flat over A, and $\phi : \mathscr{E}_A \to \mathscr{E}$ inducing an isomorphism $\mathscr{E}_A \otimes_A k \simeq \mathscr{E}\}/\simeq$.

Prorepresentability and Hulls

Definition 4 (Prorepresentability)**.** *Given $F : Art(\Lambda, k) \to Sets$, let $\hat{Art}(\Lambda, k)$ be the category of complete local noetherian Λ-algebras and $\hat{F} : \hat{Art}(\Lambda, k) \to Set$ defined by $\hat{F}(R) = \varprojlim F(R/\mathfrak{m}^n)$. A functor F is called prorepresentable if $\hat{F}$ is representable.*

Warning! If we start with a global moduli problem, then $\hat{F}$ is not necessarily obtained by simply considering the families over R. This is the issue of effectivizability, see next week.

Definition 5 (Smooth)**.** *Given F, F' functors $Art(\Lambda, k) \to Set$ and a natural transformation $f : F \to F'$, then f is smooth (formally smooth) if for all surjections $A \to B$ in $Art(\Lambda, k)$, the map $F(A) \to F(B) \times_{G(B)} G(A)$ is surjective.*

Recall that T_F, the tangent space of F is $F(k[\epsilon])$.

Notation: Given $R \in \hat{Art}(\Lambda, k)$ denote $h_R : \hat{Art}(\Lambda, k) \to Set$ the functor of points of $\operatorname{Spec} R$, so $H_R(R) = \hom(R, R')$ and $\bar{h}_R$ is the restriction to $Art(\Lambda, k)$.

Definition 6 (Hull)**.** *Let F be a predeformation functor, a pair (R, η) with $\eta \in \hat{F}(R)$ is a hull for F if*

1. *$\eta : \bar{h}_R \to F$ is smooth*
2. *$T_{\bar{h}_R} \to T_F$ is an isomorphism*

Proposition 2. *If (R, η) and (R', η') are hulls for F, then they are isomorphic.*

Left as an exercise.

Definition 7 (Small Thickening)**.** *A surjective map $f; A \to B$ in $Art(\Lambda, k)$ is a small thickening if $\ker f$ is isomorphic to k, or equivalently, $\mathfrak{m}_A \ker f = 0$ and $\ker f$ is principal.*

Remark: it is easy to check that any surjection in $Art(\Lambda, k)$ can be factored as a sequence of small thickenings.

Given $A' \to A, A'' \to A$, then we get $(*)$ $F(A' \times_A A'') \to F(A') \times_{F(A)} F(A'')$.

Theorem 3 (Schlessinger's Criterion)**.** *If F is a predeformation functor, consider the following conditions:*

H1 *$(*)$ is surjective whenever $A'' \to A$ is surjective. (Equivalently, Small thickening)*

H2 $(*)$ *is bijective whenever* $A'' = k[\epsilon]$, $A = k$.

H3 T_F *is finite-dimensional*

H4 $(*)$ *is bijective whenever* $A' = A''$ *and both maps are surjective. (equivalently, small thickening)*

Then $(H1) - (H3)$ *is equivalent to* F *having a hull, and* $(H1) - (H4)$ *is equivalent to* F *being prorepresentable.*

Vakil

The Space that Wanted to be a Scheme

Let $\mathscr{X} \to B$ be a nice family of objects parametrized by B. A moduli space $\mathscr{M}$ means that every family of nice objects $\mathscr{X} \to B$ corresponds to a map $B \to \mathscr{M}$.

In question: The moduli space of smooth genus 3 curves $\mathscr{M}_3$.

So $\mathscr{X} \to B$ is a smooth, projective morphism of relative dimension 1 with geometrically connected fibers all of which are genus 3.

Why should this be a scheme?

Clues: (Martin) The tangent space: given $[C] \in \mathscr{M}_3$, we can find $T_{[C]}$. At each point this is a 6 dimensional vector space. Soon (Brian) we will have the entire deformation space. So then $\mathscr{M}_3$ is formally smooth. So what is a vector bundle on $\mathscr{M}_3$?

$$\begin{array}{ccc} \mathscr{C} & & V \\ \downarrow & V_B = \pi^* V \swarrow & \downarrow \\ B & \xrightarrow{\pi} & \mathscr{M}_3 \end{array}$$

So a vector bundle on $\mathscr{M}_3$ should be a recipe for giving, a vector bundle on B for any family $\mathscr{C} \to B$ which behaves well under pullback.

e.g., the Hodge bundle

$$\begin{array}{ccc} \mathscr{C} & & \mathbb{E}^3 \\ \downarrow \pi & \pi_* \Omega^1_C \swarrow & \downarrow \\ B & \longrightarrow & \mathscr{M}_3 \end{array}$$

So now, we take $\mathscr{H}_3 \subseteq \mathscr{M}_3$, some genus 3 curves are hyperelliptic. (i.e., they admit 2 to 1 covers of $\mathbb{P}^1$. $\mathscr{H}_3$ is a 5-dimensional Cartier Divisor, so it is locally cut out by a single equation. In fact, this holds in general, if you have a family $\mathscr{C} \to B$, there is a closed immersion $H \to B$ locally cut out by a single equation such that $C|_H$ (the completion of the fiber square) is a family of hyperelliptic curves and that H is the largest closed immersion with this property. (Obtain H by completing the square with $\mathscr{H}_3 \to \mathscr{M}_3$ and $B \to \mathscr{M}_3$.

There is a scheme $\mathscr{M}_3[100]$ which are genus 3 curves with "level 100 structure" which is a legitimate scheme, and we should have a finite morphism to $\mathscr{M}_3$. In fact, an étale morphism.

We now want to, given X, Y mapping to $\mathscr{M}_3$, construct $X \times_{\mathscr{M}_3} Y$. We get two families of curves on $X \times Y$, and so $X \times_{\mathscr{M}_3} Y$ parametrizes isomorphisms between the two families.

The Isom Functor

Let B be a base and $\mathscr{C}_1, \mathscr{C}_2 \to B$ two families.

Theorem 4 (Grothendieck). $Isom_B(\mathscr{C}_1, \mathscr{C}_2)$ *is a scheme if both families are projective and flat.*

We now interpret $\mathscr{M}_{3,1} \to \mathscr{M}_3$ where $\mathscr{M}_{3,1}$ is the moduli space of genus 3 curves with 1 marked point.

This map should be a recipe for taking a family of marked genus 3 curves to a family of genus 3 curves with no marked points.

This morphisms is even projective and smooth. Why? Take $B \to \mathscr{M}_3$ and complete the fiber square, well, $B \to \mathscr{M}_3$ is the same as a family $\mathscr{C} \to B$ and so $\mathscr{C}$ complete the square.

Unfortunately, $\mathscr{M}_3$ is not a scheme!

Max said that schemes are sheaves in the Zariski Topology, and that means that if X is a scheme, we fix a scheme Y and $U \mapsto \hom(U, X)$ is a sheaf on Y.

Let Y be a pair of $\mathbb{P}^1$'s glued together over 2 points. Fix a curve C of genus 3 with a nontrivial automorphism σ. We will use this to create two families of curves over Y.

The first will be $C \times Y \to Y$. The second is done by taking the trivial families on $\mathbb{P}^1$ and gluing by applying σ on the two points. This will be called the Moebius Strip family. Erasing either node trivializes the Moebius Strip, so this cannot be a scheme!

Grothendieck Topologies

We want to think of the category as open sets of a space X. $U \to X$ maps.

Take the additional data of coverings with three axioms (see Lieblich 2).

We now look at $\mathbb{A}^1_{\mathbb{C}} \sim \mathbb{C}$ with the classical topology, and it has coordinate t, then $\sqrt{t}$ is a function on some small enough open set U. $\sqrt[3]{t-1}$ makes sense on some V. And in fact $\sqrt{t} + \sqrt[3]{t-1}$ makes sense on $U \cap V$.

Take a map $\operatorname{Spec} \mathbb{C}[u] \setminus \{0\} \to \operatorname{Spec} \mathbb{C}[t] \setminus \{0\}$ by $u^2 = t$ and this is an étale open set, so this shows that the étale topology is closer to the classical topology.

How to save $\mathscr{M}_3$. The problem was the automorphisms.

Example: $x^7 + y^7 + z^7 = 0$ and $a^7 + b^7 + c^7 = 0$. They are isomorphic, but there are MANY isomorphisms.

The Category of Families of Genus 3 Curves

This is a big groupoid, because you only have isomorphisms. We have a functor that takes a family to it's base in Schemes. We want to add morphisms that allow base change, and those will be fiber squares. This makes it no longer a groupoid.

This is an example of a category fibered in groupoids.

A category fibered in groupoids is a STACK if

1. If $Isom_X(\mathscr{C}_1, \mathscr{C}_2)$ from $Sch_X \to Sets$ is a sheaf in the étale topology

2. "Elements glue"

Fibered products of stacks are a little more subtle than with schemes...

Definition 8 (Representable Morphism)**.** *A morphism of stacks $\mathscr{M} \to \mathscr{M}'$ is representable if for all $X \to \mathscr{M}'$ with X a scheme, the fiber product $X \times_{\mathscr{M}'} \mathscr{M}$ is a scheme.*

Any notion preserved by base chance makes sense for representable morphisms.

Exercise: If $\mathscr{M}$ is a stack, then the diagonal $\Delta_{\mathscr{M}} : \mathscr{M} \to \mathscr{M} \times \mathscr{M}$ is representable iff every morphism from a scheme to $\mathscr{M}$ is representable.

Hint:

$$\begin{array}{ccc} U \times_{\mathscr{M}} V & \longrightarrow & U \times_Y V \\ \downarrow & & \downarrow \\ \mathscr{M} & \longrightarrow & \mathscr{M} \times_Y \mathscr{M} \end{array}$$

Definition 9 (Deligne-Mumford Stack)**.** *A stack $\mathscr{M}$ is of Deligne-Mumford type if*

1. $\Delta_{\mathscr{M}}$ is representable, quasi-compact and separated

2. There is an étale cover by a scheme