1 Hacking

Constructing Surfaces of General Type by Deformation Theory

Motivation: Let $k = \mathbb{C}$. Then curves C have only one topological invariant, the genus g.

Surfaces: We consider the underlying topological 4-manifold. Then $Q = \cup : H^2(X,\mathbb{Z}) \times H^2(X,\mathbb{Z}) \to H^2(X,\mathbb{Z}) = \mathbb{Z}$ is a unimodular symmetric bilinear form.

Freedman: Assume that $\pi_1(X) = 0$. Then X is uniquely determined up to homeomorphism by Q.

Donaldson: $\exists$ a topological 4 manifold with infinitely many non-isomorphic differentiable structures (eg, elliptic surfaces) (never true for dim $\neq 4$.)

Problem: Classify topological types of surfaces of general type? (recall, a surface is of general type if $\omega_X^{\otimes N}$ defines a birational map for $N >> 0$).

Assume that $\pi_1(X) = 0$. Then Hodge Theory implies that the signature of Q is $(2h^{2,0}+1, h^{1,1}-1)$ where $h^{2,0} = h^0(K_X)$. Consider the simplest case, where $h^{2,0} = 0$. Then Q is $diag[1,-1,\ldots,-1]$ for n -1's, for some n.

$X \simeq Bl^n\mathbb{P}^2$ (homeomorphic)

(Topologist: $\mathbb{P}^2_{\mathbb{C}}\sharp\bar{\mathbb{P}}^2_{\mathbb{C}}\sharp\ldots\sharp\bar{\mathbb{P}}^2_{\mathbb{C}}$ where $\sharp$ is a connected sum and $\bar{\mathbb{P}}^2$ is $\mathbb{P}^2$ with orientation reversed.)

Severi, around 1920, asked if there is a surface X of general type such that $X \simeq \mathbb{P}^2_{\mathbb{C}}$.

Yau, in 1976, showed no. By showing that $c_1^2 = 3c_2 \Rightarrow X = B/\Gamma$ where $B \subset \mathbb{C}^2$ is the unit ball and Γ acts on B as the proper discontinuous action of a discrete group.

Theorem 1 (Barlow 1982 (A student of Miles Reid)). *There exists a surface of general type with X homeomorphic to $\mathbb{P}^2$ blown up at 8 points.*

He did this by explicit construction.

Theorem 2 (Lee and Park 2006). *There exists a surface of general type X homeomorphic to $\mathbb{P}^2$ blown up at 7 points.*

(Yesterday posted 6)

Idea of Proof:

Assume that there exists such an X. Consider the moduli space $\mathscr{M}$ of deformations of X. Then $\dim \mathscr{M} \geq h^1(T_X) - h^2(T_X)$ (because $h^0(T_X) = 0$ because X being of general type implies that there are no infinitesimal automorphisms.)

So then $h^1(T_X) - h^2(T_X) = -\chi(T_X)$.

By Riemann-Roch, if E is a vector bundle on X, then $\chi(E) = \deg(ch(E).td(X))_{\dim X}$ and in this case, $\chi(T_X) = 1/6(7c_1^2 - 5c_2) = 1/6(7*2-5*10) = -6$, and so $\dim\mathscr{M} \geq 6$.

We compactify $\mathscr{M}$ by adding points corresponding to singular surfaces at the boundary (there exists a natural way to do this using minimal model program).

To prove the theorem, we construct singular surface Y corresponding to a point of $\partial\mathscr{M}$ and proof that there exists a smoothing.

<u>Local Model</u>

Notation: We will write $\frac{1}{r}(1,a) = \mathbb{C}^2/\mu_r$ where $\mu_r \ni \mathscr{S} : (x,y) \mapsto (\mathscr{S}x, \mathscr{S}y^a)$. we always assume that $(a,r) = 1$ because that is iff it is free in codimension 1. Consider $Y = \frac{1}{n^2}(1, n\alpha - 1)$.

Smoothing: $Z = 1/n(1,-1) = (uv = w^n) \subset \mathbb{C}^3$ which corresponds to $Y = 1/n^2(1, n\alpha-1) = (uv = w^n) \subset 1/n(1,-1,a)$.

$(u = x^n, v = y^n, w = xy)$ so get $\mathscr{Y} = (uv = w^n + t) \subset 1/n(1,-1,a) \times \mathbb{C}^1_t$,

Note that K_Y is $\mathbb{Q}$-Cartier.

<u>Milnor Fibre</u>

1. Milnor fibre M' of smoothing $\mathscr{Z} = (uv = w^n + t)$ of Z.

   Brieskorn: there exists a simultaneous resolution of family of ADE singularities (after finite base change). This tells us that M' is diffeomorphic to a family of smooth manifolds, which is homotopy equivalence to $\vee_{i=1}^{n-1} S^2$.

2. The Milnor fiber of $Y = \mathscr{Y}$, and then $M' \stackrel{\mu_n}{\to} M$ étale implies that $\pi_1(M) = \mathbb{Z}/n\mathbb{Z}$ and $e(M') = ne(M)$.

So M has the homotopy type of a CW complex of real dimension 2 via Morse Theory. Thus, M is a rational homology ball, ie, $H_*(M,\mathbb{Q}) = H_*(B,\mathbb{Q})$.

From the Mayer-Vietoris sequence, we get $e(X) = e(X^*) + e(M) - e(\partial M)$, $e(Y) = e(Y^*) - e(C) - e(\partial M)$, and $e(C) = e(M) = 1$, so $e(X) = e(Y)$ where C is the cone.

Now we must check π_1.

$\partial M = S^3/(\mathbb{Z}/n^2\mathbb{Z})$, the lens space, and we have $\pi_1(\partial M) \to \pi_1(M)$ surjective and by van Kampen's Theorem, we get $\pi_1(X) = \pi_1(X^*) *_{\pi_1(\partial M)} \pi_1(M)$ which is surjected onto by $\pi_1(X^*) = \pi_1(Y^*)$, and so it is enough to show that $\pi_1(Y^*) = 0$.

Smoothability:

Let X be a surface with isolated singularities and $L_X = L^*_{X/k}$ the cotangent complex. Then $0 \to I \to A' \to A \to 0$ exact, specifically, $0 \to kt^{n+1} \to k[t]/(t^{n+2}) \to k[t]/(t^{n+1}) \to 0$ and X lying over $\operatorname{Spec} A$. Can we extend X to $\operatorname{Spec} A'$?

The obstruction lies in $\operatorname{Ext}^2(L_X, \mathscr{O}_X)$, if it is zero, then the extensions form a torsor under $\operatorname{Ext}^1(L_X, \mathscr{O}_X)$.

Local to Global

$H^p(\mathscr{E}xt^q) \Rightarrow \operatorname{Ext}^{p+q}$ spectral sequence takes us to an exact sequence

$$0 \to H^1(\mathscr{E}xt^0) \to \operatorname{Ext}^1 \to H^0(\mathscr{E}xt^1) \to H^2(\mathscr{E}xt^0) \to \operatorname{Ext}^2$$

Which gives

$$0 \to H^1(T_X) \to \operatorname{Ext}^1(L_X, \mathscr{O}_X) \to H^0(\mathscr{E}xt^1(L_X, \mathscr{O}_X)) \to H^2(T_X) \to \operatorname{Ext}^2(L_X, \mathscr{O}_X)$$

Claim: $H^2(T_X) = 0$ implies that every infinitesimal deformation of singularities is induced by a deformation of X.

Proof: the first order is by the last long exact sequence

higher order: the spectral sequence gives $\operatorname{Ext}^2(L_X, \mathscr{O}_X) = H^0(\mathscr{E}xt^2(L_X, \mathscr{O}_X))$, so if $\exists$ a local lift, then there exists a global lift. Now liftings are torsors under $\operatorname{Ext}^1(L_X, \mathscr{O}_X)$ which is global, and maps surjectively onto $H^0(\mathscr{E}xt^1(L_X, \mathscr{O}_X))$ which is local.

WARNING

If X is a surface of general type, then often $H^2(T_X) \neq 0$. eg, $X = X_d \subset \mathbb{P}^3$, $d \geq 5$ implies that $H^2(T_X) \neq 0$. (Exercise)

Reason: $H^2(T_X) = H^0(\Omega_X \otimes \omega_X)^*$ by Serre Duality and ω_X ample.

However, X may still have unobstructed deformations. eg, $X = X_d \subset \mathbb{P}^3$, the embedded deformations of $X \subset \mathbb{P}^3$ modulo isomorphism surjects onto the deformations of X modulo isomorphism for $d \neq 4$ (Exercise)

The Construction

First thing, Y is rational. There exist singular rational surfaces with ample canonical bundle.

If C is cubic, p, q, r flexes, B is a conic and A is a line in $\mathbb{P}^2$. Consider the pencil of cubics generated by $A + B, C$. Blow up the base points p, q, r three times to get an elliptic fibration $p : Z \to \mathbb{P}^1$

The degenerate fibers are $\tilde{E}_6, \tilde{A}_1$ and 2 nodes for C general.

Blow up 18 times to get $g : \tilde{Z} \to Z$.

Then we contract 5 chains of $\mathbb{P}^1$'s where a chain of $\mathbb{P}^1$'s with self intersection ≤ -2, contracts to a cyclic quotient singularity.

General Type

$K_{\tilde{Z}} = g^* K_Z + E$, with E effective and g-exceptional.

$K_{\tilde{Z}} = h^* K_Y - F$ where F is effective and h-exceptional.

Then $h^* K_Y = h^* K_Z + E + F = E + F - f$ where f is a fiber of $\tilde{Z} \to \mathbb{P}^1$ and so $h^* K_Y \sim D$ is effective, now check that $h^* K_Y$ is nef, ie, $h^* K_Y . C \geq 0$ for all C (We only need to check $C = \operatorname{Supp} D$).

So then $K_Y^2 = 2$ from Noether's formula. ($c_1^2 + c_2 = 12\chi(\mathscr{O}_Y), K_Y^2 + e(Y) = 12\chi(\mathscr{O}_X)$ and $h^1(\mathscr{O}_Y) = h^2(\mathscr{O}_Y) = 0$ since Y is rational).

So why does $H^2(T_X) = 0$?

Lemma 1. *Y is a surface with cyclic quotient singularities and $\pi : \tilde{Z} \to Y$ is the minimal resolution with E the exceptional locus. IF $H^2(T_{\tilde{Z}}(-\log E)) = 0$ then $H^2(T_Y) = 0$. (That is, the vector fields tangent to E)*

Serre duality gives that $H^2(T_{\tilde{Z}}(-\log E)) = H^0(\Omega_{\tilde{Z}}(\log E) \otimes \omega_{\tilde{Z}})^*$, so now we use the elliptic fibration $\tilde{Z} \to \mathbb{P}^1$ to get vanishing.

2 Conrad

Discrete Galois Modules

Let F be a field and F_s/F a fixed separable closure with Galois group $G_F = \mathrm{Aut}(F_S/F) = \varprojlim Gal(F'/F)$ is a compact, profinite group.

Krull Correspondence says that closed subgroups correspond to intermediate fields and that open subgroups correspond to finite subextensions

Example 1. *V a commutative group scheme of finite type over F*

Then G_F acts on $M = V(F_S)$ (an abelian group)

This acts on $\mathrm{Spec}\, F_S \to V$, any such map factors as $\mathrm{Spec}\, F_s \to \mathrm{Spec}\, F' \to V$ for a finite Galois extension

V is quasiprojective, so $V \hookrightarrow \mathbb{P}^N_F$ and it acts on the homogeneous coordinates.

G_F action on any $m \in V(F') \subseteq V(F_S) = M$ has stabilizer $Gal(F_S/F') \subseteq G_F$ an open subgroup.

Example 2. *$X \to \mathrm{Spec}\, F$ separated and of finite type, then $\ell \neq \mathrm{char}\, F$, $H^i_{et}(X_{F_s}, \mathbb{Z}/\ell^n\mathbb{Z})$ has a G_F action with open stabilizers.*

Definition 1 (Discrete Galois Module)**.** *A discrete G_F-module is a G_F-module M such that every element $m \in M$ has an open stabilizer in G_F.*

Example 3. *A G_F-module M with $|M| < \infty$ is discrete iff G_F acts on M through a finite quotient $Gal(F'/F)$ (IE, $G_{F'} \subseteq G_F$ open acts trivially on M)*

Remark 1. *If Γ is any profinite group, then we can make the same discussion. Examples are $\Gamma = \mathbb{Z}_p, GL_n(\mathbb{Z}_p)$ and $\pi_1^{et}(X, x)$.*

Example 4. *E/F is an elliptic curve, $N \in \mathbb{Z}^+$ and $\mathrm{char}\, F \not| N$. Then $E[N] := E[N](F_s) \simeq \mathbb{Z}/N\mathbb{Z} \times \mathbb{Z}/N\mathbb{Z}$ with G_F acting through $Gal(F(E[N])/F)$ extension generated by the coordinates of the N-torsion points.*

$\rho_{E,N} : G_F \overset{cont}{\to} \mathrm{Aut}(E[N]) \simeq GL_2(\mathbb{Z}/N\mathbb{Z})$.

Consider $N = p^r$ where p is a prime not equal to $\mathrm{char}\, F$

Fact: $E[p^r] \simeq \mathbb{Z}/p^r\mathbb{Z} \times \mathbb{Z}/p^r\mathbb{Z}$.

If $F = \mathbb{C}$ then $E \simeq \mathbb{C}/\Lambda$ so $E[N] \simeq \frac{1}{N}\Lambda/\Lambda \simeq \Lambda/N\Lambda$ which is surjected onto by $\Lambda/Nd\Lambda \simeq E[Nd]$, and so $E[p^{r+1}] \to E[p^r]$ is surjective.

So we can set G_F acting on each thing in a way that is compatible. So we are going to deform this compatible system, $\rho_{E,N}$.

Define $T_p(E) =$ the p-adic Tate module, which is $\varprojlim E[p^r] = \mathbb{Z}_p \times \mathbb{Z}_p$. So if $F = \mathbb{C}$, $E \simeq \mathbb{C}/\Lambda$, then $T_pE = \varprojlim \Lambda/p^r\Lambda \simeq \Lambda \otimes_{\mathbb{Z}} \mathbb{Z}_p = H_1(E, \mathbb{Z}_p)$.

So now we have $\rho_{E,p^\infty} : G_F \to GL_2(\mathbb{Z}_p) \subseteq M_2(\mathbb{Z}_p)$ open, and usually the kernel isn't open. Now $T_p(E)$ is a Galois Module, but not a discrete one.

Arithmetic Application: Let F be a finite field of order q, and choose $\ell \neq \mathrm{char}\, F$, so we get $\rho_{E,\ell^\infty} : G_F \to GL_2(\mathbb{Z}_\ell) = \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell E)$. Let ϕ be the Frobenius element in G_F. Then the characteristic polynomial of ϕ is $X^2 - a_E X + q$ where $a_E = |E_F| - (q+1) \in \mathbb{Z} \subset \mathbb{Z}_\ell$.

Example 5. *Let E be an elliptic curve over $\mathbb{Q}_p$ and E' another one. Define them by $y^2 = x^3 + ax + b$ and $y^2 = x^3 + a'x + b'$. Look at the $G_{\mathbb{Q}_p}$-action on $E[p^7], E'[p^7]$. Then you get two maps $G_{\mathbb{Q}_p} \rightrightarrows GL_2(\mathbb{Z}/p^7\mathbb{Z})$, and suppose that $|a' - a|, |b' - b| << 1$, then $\rho_{E,p^7} \simeq \rho_{E',p^7}$ as $G_{\mathbb{Q}_p}$-modules.*

So then we have $\rho_{E,p^\infty}, \rho_{E,p^\infty} : G_{\mathbb{Q}_p} \to GL_2(\mathbb{Z}_p)$ both induce the same map to $GL_2(\mathbb{Z}/p\mathbb{Z})$. "two p-adic deformations" of the same induced p^7 representation.

Cohomology

Let Γ be a profinite group (eg G_F). Define Mod_Γ to be the category of discrete Γ-modules. ($\neq \mathbb{Z}[\Gamma]$-modules)

Exercise 1. *Mod_Γ has enough injectives (follow Tohoku)*

$Mod_\Gamma \to Ab$ by $M \mapsto M^\Gamma = \{m \in M | \gamma m = m \text{ for all } \gamma \in \Gamma\}$.

Definition 2 (Cohomology)**.** *$H^*(\Gamma, -) : Mod_\Gamma \to Ab$ are the derived functors of $(-)^\Gamma$.*

Remark: We can compute this using continuous cochains.

Example 6. *$H^1(\Gamma, M) = Z^1(\Gamma, M)/B^1(\Gamma, M)$ where $B^1(\Gamma, M) = \{\Gamma \to M$ continuous sending $\gamma \mapsto \gamma m_0 - m_0$ for some $m_0 \in M\}$. This factors through $\Gamma/stab_\Gamma(m_0)$.*

$Z^1(\Gamma, M) = \{c : \Gamma \to M | c(\gamma_1\gamma_2) = \gamma_1 c(\gamma_2) + c(\gamma_1),\ c$ cont$\}$.

Say that Γ acts trivially on M, then $B^1(\Gamma, M) = 0$ and $Z^1(\Gamma, M) = \{$continuous homomorphisms $\Gamma \to M\}$.

Remark 2. *Given $\varphi : \Gamma \to \Gamma'$ continuous, then get $Mod_{\Gamma'} \to Mod_\Gamma$ and for $M' \in Mod_{\Gamma'}$, we have $(M')^{\Gamma'} \subset (M')^\Gamma$.*

This induces $H^*(\Gamma', M') \to H^*(\Gamma, M')$ by composition with φ on the level of cochains.

Example 7. *F'/F a field extension, say $\mathbb{Q}_p/\mathbb{Q}$, then fix compatible separable closures and get an induces $G_{F'} \to G_F$ continuous and well defined up to conjugation, and $H^*(\Gamma', M') \to H^*(\Gamma, M')$ is invariant under conjugation by Γ'.*

Thus, $H^*(G_F, M) \to H^*(G_F, M)$ is canonical, and often we just say $H^*(F, M) \to H^*(F', M)$, it's like the pullback map with respect to $\operatorname{Spec} F' \to \operatorname{Spec} F$.

Example 8. *Take $F = \mathbb{Q}$ and look at $H^1(G_\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = \hom_{cont}(G_\mathbb{Q}, \mathbb{Z}/2\mathbb{Z}) = \{\mathbb{Q}, \mathbb{Q}(\sqrt{d}) | d \in \mathbb{Z} \setminus \{0\}$ square free$\}$*

This is not finite dimensional over $\mathbb{Z}/2\mathbb{Z}$. We want to work with a quotient of $G_\mathbb{Q}$ subject to restricted ramification.

Let F be a number field ($[F : \mathbb{Q}] < \infty$) then $\operatorname{Spec} \mathscr{O}_F$ is "like" an algebraic curve. What should replace G_F? Take F'/F finite, then get $\operatorname{Spec} \mathscr{O}_{F'} \to \operatorname{Spec} \mathscr{O}_F$ and let Σ be a finite set of ramified primes. Want to consider F'/F ramified $\subset \Sigma =$ the fixed finite set of maximal ideals of $\mathscr{O}_F$, ie, we want to replace G_F with $\pi_1^{et}(\operatorname{Spec} \mathscr{O}_{F,\Sigma})$

ie, $G_{F,\Sigma} = Gal(F_\Sigma/F)$ composition of F'/F finite and unramified outside Σ.

Algebraic Number Theory $(+\epsilon)$ implies

Theorem 3 (Tate)**.** *If M is a finite discrete $G_{F,\Sigma}$-module, then $H^i(G_{F,\Sigma}, M)$ are finite for all i and vanish for $i > 2$ so long as $|M|$ is odd.*

If instead, $[L : \mathbb{Q}_p] < \infty$, then $H^i(G_L, M)$ are finite for M a finite discrete G_L-module and are 0 for $i > 2$.

Example 9. *$F = \mathbb{Q}$, $\Sigma = \{2, 3, 7\}$, then $H^1(G_{\mathbb{Q},\Sigma}, \mathbb{Z}/2\mathbb{Z}) = \{\mathbb{Q}(\sqrt{d})|$ for $d|42\}$, which is finite.*

<u>Deformations</u>

Motivation: Hida constructed certain representations $\rho : G_{\mathbb{Q},\Sigma} \to Gl_2(\mathbb{Z}_p[[x]])$ such that under $x \mapsto (1+p)^k - 1$ for $k \geq 2$ gave interesting representations $\rho_k : G_{\mathbb{Q},\Sigma} \to GL_2(\mathbb{Z}_p)$.

Now, fix $\bar{\rho} : \Gamma \to GL(V_0)$ a finite dimensional representation over a finite field of a profinite group.

Remember, $GL(V_0) \simeq GL_N(k)$

$\hat{\varphi}_k =$ the complete local noetherian rings with residue field k (= coeff ring $\Lambda = W(k)$) (If $k = \mathbb{F}_p$, then $\Lambda = \mathbb{Z}_p$.)

A lifting of $\bar{\rho}$ to $A \in \hat{\varphi}_k$ is a pair (V_A, θ) where V_A is a finite free A-module equipped with a continuous $\rho : \Gamma \to GL(V_A)$ and $\theta : V_A/\mathfrak{m}_A V_A \simeq V_0$ as $k[\Gamma]$-module.

Say $(V_A, \theta) \sim (V'_A, \theta')$ if $\exists V_A \simeq V'_A$ as $A[\Gamma]$-modules such that mod $\mathfrak{m}_A$ carries θ to θ'. (ie, respects identification with V_0)

A deformation of $\bar{\rho}$ to A is an equivalence class of lifts.

Matrix Meaning: $\rho : \Gamma \to GL_N(k)$.

Lifting: $\rho : \Gamma \to GL_N(A)$ continuous such that $\rho \mod \mathfrak{m}_A = \bar{\rho}$ and $\rho \sim \rho'$ corresponds to $\rho = M\rho' M^{-1}$ with $M \in GL_N(A)$ and $M \equiv 1 \mod \mathfrak{m}_A$.

WARNING: If $E \to S$ (S is a p-adic variety) is a family of elliptic curves, for all $s \in S(\mathbb{Q}_p)$ set $\rho_s : G_{\mathbb{Q}_p} \to GL_2(\mathbb{Z}_p)$ from E, but these don't come from a single representation $G_{\mathbb{Q}_p} \to GL_2(\mathbb{Z}_p[[x_1, \ldots]])$

Definition 3. $\mathrm{Def}_{\bar{\rho}} : \hat{\varphi}_k \to$ *Set takes* A *to the set of deformations of* $\bar{\rho}$ *to* A*.*

This is easily seen to be a covariant functor using $V_A \to A' \otimes_A V_A$.
Note: These are deformations, not liftings. EQUIVALENCE CLASSES.

Exercise 2. $\mathrm{Def}_{\bar{\rho}}(k[\epsilon]) = H^1(\Gamma, \mathrm{End}_k(V_0))$ *with gamma acting by conjugation on linear maps.*

"Proof": $\bar{\rho} : \Gamma \to GL_N(k)$, $\rho : \Gamma \to GL_N(k[\epsilon])$ and $\rho(\gamma) = (1 + \epsilon c(\gamma))\bar{\rho}(\gamma)$. ρ continuous liftings correspond to $C \in Z^1(\Gamma, \mathrm{End}(V_0))$ and $\rho \sim \rho'$ corresponds to $c - c' \in B^1(\Gamma, \mathrm{End}(V_0))$.

Theorem 4 (Mazur)**.** *If* $\dim H^1(\Gamma, \mathrm{End}\, V_0) < \infty$*, then* $\mathrm{Def}_{\bar{\rho}}$ *satisfies Schlessinger's criteria (H1) to (H3).*
If $\mathrm{End}_\Gamma(V_0) = k$ *(eg* $\bar{\rho}$ *is absolutely irreducible) then (H4) also holds, so get a universal deformation* $\bar{\rho}^{univ} : \Gamma \to GL_N(\mathscr{R}_{\bar{\rho}}^{univ})$*.*

IE, given $\bar{\rho} : \Gamma \to GL_N(k)$ and a lifting $\rho : \Gamma \to GL_N(A)$, then there exists a unique $\mathscr{R}_{\bar{\rho}}^{univ} \to A$ such that it carries $\bar{\rho}^{univ}$ to ρ up to 1-unit matrix conjugation.

Example 10. $\Gamma = G_{\mathbb{Q},\Sigma}$*, want to impose more conditions.*