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Why should people care about Deformation Theory?

We will be working over $\mathbb{C}$. And we take any kind of object and consider one, X. The question is, imprecisely, how can we deform X. Infinitesimally generally means roughly power series.

$$\begin{array}{ccc} X_0 & \xrightarrow{\text{inc}} & X \\ \downarrow & & \downarrow \text{nice} \\ \operatorname{Spec}\mathbb{C} & \xrightarrow{\text{inc}} & \operatorname{Spec} A \end{array}$$

In algebraic geometry, nice means flat, at the least.

Sample application: Suppose scheme $\mathscr{M}$ which parametrizes your objects. "fine moduli space" means that "every time there is a nice family of objects, then that corresponds to a map to $\mathscr{M}$, and vice versa" and we also need this to be a functor so that we get base change right.

If you aren't so lucky as to have a fine moduli space, then look for a coarse moduli space, which isn't perfect, but in a precise sense, it is the closest you can get. Otherwise, Deligne-Mumford stacks.

Questions with moduli spaces: pick a point (object) and look near that point. Find the formal nbhd, is it smooth? unibranch?

Helpful thing is then that if you can prove something about an object X_0 and show that it holds under deformation, then you have it for many other objects.

Example(Deligne-Mumford '69): The moduli space of genus g curves is irreducible. The space is called $\mathscr{M}_g$. The method used is to prove that it is connected and smooth. They solved the "harder" problem of compactifying the space.

So now, we take a moduli space which is a scheme, and let X be an object of the sort with $(R, \mathfrak{m})$ its local ring. We want to look at $\hat{R}$ and we build it level by level.

Example: Take $\mathscr{M} = V(y^2 - x^3)$. So $R = \mathbb{C}[x,y]/(y^2-x^3)$. We get for the Tangent Space $\mathbb{C}[x,y]/(x,y)^2 \leftarrow \mathbb{C}[x,y]/ < (x,y)^3, y^2 > \leftarrow \mathbb{C}[x,y]/ < (x,y)^4, y^2 - x^3 >$. There is an obstruction space which, if zero, then the space is smooth.

There are three relevant spaces: aut/def/ob, and so if X is a smooth variety, Def $= H^1(X, \mathscr{T}_X)$, ob$=H^2(X, \mathscr{T}_X)$ and $H^0(X, \mathscr{T}_X)$ is aut.

Exercise: $\mathscr{M}_g$ for $g \geq 2$, and $\bar{\mathscr{M}}_g$. What if X is singular? $\operatorname{Ext}^1(\Omega_X, \mathscr{O}_X) = def$, $ob = \operatorname{Ext}^2$ and $aut = H^0$. Fix X and deform E a vector bundle on X. Aut, the infinitesimal deformations, are $\hom(E,E)$.

(Note: Derived Categories are lurking here)

Example: Manifold in $\mathbb{CP}^n$, assume $\pi : X \to \mathbb{P}^n$ is an embedding of a nonsingular projective variety. $def(X \to \mathbb{P}^n) = H^0(X, \mathscr{N}_{X/\mathbb{P}^n})$, and to do things with it, we get the normal bundle sequence: $0 \to \mathscr{T}_X \to \pi^*\mathscr{T}_{\mathbb{P}^n} \to \mathscr{N}_{X/\mathbb{P}^n} \to 0$, and we get the long exact sequence on homology.

$0 \to H^0(\mathscr{T}_X) \to H^0(\pi^*\mathscr{T}_{\mathbb{P}^n}) \to H^0(\mathscr{N}_{X/\mathbb{P}^n}) \to H^1(\mathscr{T}_X) \to H^1(\pi^*\mathscr{T}_{\mathbb{P}^n}) \to H^1(\mathscr{N}_{X/\mathbb{P}^n}) \to \ldots$ which is also $0 \to \operatorname{Aut}(X) \to def\pi$(deform the embedding)$\to def(\pi : X \to \mathbb{P}^n) \to def(X) \to \operatorname{ob}(\pi) \to \operatorname{ob}(\pi : X \to \mathbb{P}^n) \to \ldots$.

This generalizes, we can replace X by something singular and $\mathbb{P}^n$ by Y.

Max Lieblich

What is a Moduli Space?

Well, objects seem to come in types

1. Varieties
2. Curves of genus g
3. Line Bundles on X
4. Maps between X and Y
5. Closed Subschemes of X (often $X = \mathbb{P}^n$)

6. Subspaces of a vector space

So {pts of $\mathscr{M}_i$} $\leftrightarrow$ {objects of flavor i}
Q: How should a scheme be described?

1. "Absolute", ie, give affine charts and glue them together

2. "Relative"

Analogy: Functions. Question: How should an element $f \in C^\infty([0,1])$ be described.

There is the absolute method of just saying what f is at each point. The relative way uses the pairing $g \in C^\infty$ and takes $\int_0^1 fg \in \mathbb{R}$. So this gives $L_f : C^\infty \to \mathbb{R}$ by $g \mapsto \int fg$, and there is a theorem $L : C^\infty \to \hom(C^\infty, \mathbb{R})$ is an injective linear transformation.

We can do the same thing in any category $\mathcal{C}$. We take the pairing $X, Y \mapsto \hom(X, Y)$, a set. So $Y \mapsto \hom(Y, X)$ called $h_X(Y)$, gives us a functor $\mathcal{C} \to Func(\mathcal{C}^\circ, Sets)$, and injectivity corresponds to being fully faithful.

$Z \to Y$ gives $h_X(Y) \to h_X(Z)$ by composition. h_X is a contravariant functor from $\mathcal{C}$ to sets, or a covariant functor from $\mathcal{C}^\circ$ to Sets.

Example: $\mathcal{C} = Sch_\mathbb{Z}$, ie, all schemes. $h_{\mathbb{A}^1}(Y) = \Gamma(Y, \mathscr{O}_Y)$ and $Z \to Y$ gives the pullback map on functions.
$h_{\mathbb{G}_m}(Y) = \Gamma(Y, \mathscr{O}_Y^*)$ and $h_{\mathbb{P}^n}(Y) = \{\mathscr{O}_Y^{n+1} \to \mathscr{L}$ surjective, with $\mathscr{L}$ invertible on $Y\}/\simeq$.
Now, $Func$ is a category, with objects being functors and morphisms being natural transformations.
Analogue of Bumps is

Lemma 1 (Yoneda's Lemma)**.** *$h : \mathcal{C} \to Func(\mathcal{C}^\circ, Sets)$ is fully faithful.*

eg, $h_{\mathbb{A}^1} \to h_{\mathbb{P}^n}$ will always come from a unique map of schemes $\mathbb{A}^1 \to \mathbb{P}^n$.

Proof. Faithful: Let $f, g : X \to Y$ in $\mathcal{C}$, then $h_f, h_g : h_X \to h_Y$. Show that $h_f = h_g$ implies $f = g$. If $h_f(T) : h_X(T) \to h_Y(T)$ for all $T \in \mathcal{C}$ and $h_f(X) = h_X(X) \to h_Y(X)$, that is $\mathrm{id}_X \mapsto f$.
$h_g(X) : h_X(X) \to h_Y(X)$ takes $\mathrm{id}_X \mapsto g$.
Full: Given $F : h_X \to h_Y$, show that $F = h_f$ for some $f : X \to Y$, take $\phi : T \to X$. Then $h_X(X) \to h_X(T)$ takes $\mathrm{id}_X \to \phi$.
DIAGRAM CHASE □

h_X is called the functor of points of X.
Example: $\mathbb{G}_m \to \mathbb{G}_m$ by $x \mapsto x^2$ works on all points and describes a natural transformation $h_{\mathbb{G}_m} \to h_{\mathbb{G}_m}$.
Philosophy: Functor=generalized space and schemes are a distinguished class of spaces.
Instead of $F : \mathcal{C}^\circ \to Sets$ think of F and take its internal structure as $F(T)$.
Exercise of Yoneda Type: $\exists$ a natural bijection $\hom(h_T, \mathscr{F}) \simeq F(T)$

Definition 1 (Representable Functor)**.** *A functor F is representable if there exists $X \in \mathcal{C}$ such that $F \simeq h_X$.*

Anything we can do with sets, we can do with functors to sets.
eg: $F, G \to H$ lets us create fiber product $F \times_H G(T) = F(T) \times_{H(T)} G(T)$. This works!
Q: What is the functor of points of M_i?
Look at $M_0(T) = \{X \to T |$finite presentation, flat, geometrically integral fibers$\}/\simeq$.
$M_1(T) = \{C \to T |$proper, smooth, finite presentation, fibers are smooth of genus $g\}/\simeq$.
$M_2(T) = \{\mathscr{L}$ invertible on $X \times T\}/\simeq$.
$M_3(T) = \{\hom_T(X \times T, Y \times T)\}$
$M_4(T) = \{Z \to X \times T$ over T, Z is T-flat closed immersion of $X \times T$, finite presentation$\}/\simeq$.
$M_5(T) = \{W \subset V \otimes \mathscr{O}_T$ such that coker is locally free$\}$
Martin Olsson
Goal for first week is examples, second week is to get to cotangent complexes.
Motivation: $k = \bar{k}$, X/k is a scheme of finite type over k, and $x \in X(k)$.

Definition 2 (Tangent Space)**.** *The tangent space of X at x is the dual of the k-vector space $\mathfrak{m}/\mathfrak{m}^2$ where $\mathfrak{m} \subseteq \mathscr{O}_X$ is the maximal ideal.*

Dual Numbers. Let R be a ring, I an R-module, and $R[I]$ the ring of dual numbers. As a group, $R[I] = R \oplus I$ with $(r,i)(r',i') = (rr', r'i + ri')$

We have the inclusion of $R \to R[I]$ followed by the projection equal to id.

Remark 1: $R[I]$ is functorial in I. $g : I \to J$ induces a map $R[I] \to R[J]$ over R.

Remark 2: $I = R$, we write $R[\epsilon]$ for $R[I]$ (really should be $R[\epsilon]/\epsilon^2$)

Remark 3: Let X be a topological space and $\mathscr{O}$ is a sheaf of rings on X and I is an $\mathscr{O}$-module. We can define $\mathscr{O}[I]$.

In particular, if X is a scheme and I is a quasi-coherent $\mathscr{O}_X$-module, then we get a ringed space $X[I]$ given by $(|X|, \mathscr{O}_X[I])$.

Exercise: Show that $X[I]$ is a scheme.

Relationship with Derivations:

Let $A \to R$ be a ring homomorphism and M and R-module. An A-derivation from R to M is an A-linear map $\partial : R \to M$ such that $\partial(xy) = \partial(x)y + x\partial(y)$. This gives an R-module $\mathrm{Der}_A(R, M)$.

We look at A-algebras over R, that is, the category of pairs (C, f) where C is an A-algebra and $f : C \to R$ is a map of A-algebras.

Lemma 2. *For any A-derivation $\partial : R \to I$ the induced map $R \to R[I]$, $x \mapsto x + \partial(x)$ is a morphism in A-alg$/R$ and the induced map $\mathrm{Der}_A(R, I) \to \hom_{A-alg/R}(R, R[I])$ is a bijection.*

Proof. $s : R \to R[I]$ in $A - alg/R$ by $x \mapsto (x, \partial(x))$. A map of A-algebras corresponds to $\delta(x) = 0$ if x is in the image of A.

Composition with multiplication corresponds to $(xy, \delta(xy)) = (x, \delta(x))(y, \delta(y)) = (xy, x\delta(y) + y\delta(x))$ says that δ is a derivation. □

Remark: $f : C \to R$ surj in $A - alg/R$ and that $I = \ker f$ is square-zero. Then any section $s : R \to C$ induces an isomorphism $R[I] \simeq C$ by the short 5-lemma.

Special Case: $C = R \otimes_A R/J^2$ where J is $\ker(R \otimes_A R \to R)$ maps to R via f. If $I = J/J^2 = \Omega^1_{R/A}$ then $s : R \to C$ by $x \mapsto x \otimes 1$ induces an isomorphism $R \otimes_A R/J^2 \simeq R[\Omega^1_{R/A}]$.

This implies that $\mathrm{Der}_A(R, \Omega^1_{R/A}) \to$ sections of the diagonal map $R \otimes_A R/J^2 \to R$ is an isomorphism.

So what is the universal derivation $R \xrightarrow{d} \Omega^1_{R/A}$?

We have $\Omega^1_{R/A} = J/J^2$ and $d : R \to \Omega^1_{R/A} = J/J^2$ given by $x \mapsto x \otimes 1 - 1 \otimes x$ causes us to get the dual number $(x, 1 \otimes x - x \otimes 1)$

??? Exercise: Work This Out.

<u>The Tangent Space of a Functor</u>: Take Mod_R to be the category of finitely generated R-modules and $H : Mod_R \to Set$ a functor which commutes with finite products. IE, $H(I \times J) = H(I) \times H(J)$.

Proposition 1. *H factors canonically through the category of R-modules as $Mod_R \to R - Mod \to Set$ with the last being a forgetful functor.*

Proof. $I \times I \to I$ by $(i,j) \mapsto i + j$ gives an additive structure by $H(I) \times H(I) \simeq H(I \times I) \to H(I)$.

Thus we get a multiplicative structure $f \in R$, $\cdot f : H(I) \overset{H(\cdot H)}{\to} H(I)$.

$A - alg/R$ has finite products, so $f : C \to R$ and $f' : C' \to R$ is given by $(C, f) \times (C', f') = (C \times_R C', (x,y) \mapsto f(x) = f'(y))$

Lemma 3. *The functor $Mod_R \to A - alg/R$ by $I \mapsto (R[I], \pi : R[I] \to R)$ commutes with finite products*

Proof. Let $I, J \in Mod_R$. Then $R[I \times J] \to R[I] \times_R R[J]$ is an isomorphism over R. □

Corollary 1. *$F : A - alg/R \to Set$ such that $I, J \in Mod_R$ the map $F(R[I] \times_R R[J]) \to F(R[I]) \times F(R[J])$ is an isomorphism, then for all $I \in Mod_R$ the set $F(R[I])$ has a canonical R-module structure.*

Reason: $F(R[I])$ is the image of I under $I \mapsto R[I]$ so $Mod_R \to A - alg/R \to Sets$ commutes with finite products.

Definition 3 (Tangent Space of a Functor)**.** *Let $F : A - alg/R \to Set$ be a functor satisfying the condition in the corollary. Then the tangent space of F denoted T_F is the R-module $F(R[\epsilon])$.*

Remark: Enough that F is defined on full subcategory $\mathcal{C} \subset A - alg/R$ closed under finite products and containing $R[I]$'s. □

<u>Brian Osserman</u>

Functors of Artin Rings: Representability and Schlessinger's Criterion.

We've seen that a scheme X can be completely recovered from its functor of points (Yoneda) and under mild hypotheses, the tangent space is recovered by looking at maps $\operatorname{Spec} k[\epsilon] \to X$.

We'll look at something in between: $\operatorname{Spec} A \to X$ (with image some $x \in X$)

From a Moduli perspective, we're studying families over $\operatorname{Spec} A$ (one point) with a fixed restriction to $\operatorname{Spec} k$. These are called infinitesimal thickenings and the data obtained is the complete local ring of the moduli space at the chosen point.

Recovering Complete Local Rings: Temporary Notation: $Art(k)$ is the category of local Artin rings with residue field k (morphisms compatible with map to k). Given X a locally Noetherian scheme and $x \in X$, with $k = k(x)$. Let $F_{X,x} : Art(k) \to Set$ given by $(A \to k) \mapsto \{f : \operatorname{Spec} A \to X|$ such that $[\operatorname{Spec} k \xrightarrow{x} X] = f \circ [\operatorname{Spec} k \to \operatorname{Spec} A]\}$.

Proposition 2. *Given X locally Noetherian, $x \in X$, then the canonical map* $\operatorname{Spec} \hat{\mathscr{O}}_{X,x} \to X$ *induces* $F_{\operatorname{Spec} \hat{\mathscr{O}}_{X,x},X} \simeq F_{X,x}$*, and any complete local Noetherian ring R (with residue field k) with* $\operatorname{Spec} R \to X$ *inducing a bijection is canonically isomorphic to* $\hat{\mathscr{O}}_{X,X}$

Note: The last part anticipates prorepresentability

Proof. The first statement is equivalent to saying that any map $\operatorname{Spec} A \to X$ with image x factors uniquely $\operatorname{Spec} \hat{\mathscr{O}}_{X,x} \to X$. It's an easy exercise that it factors uniquely through $\operatorname{Spec} \mathscr{O}_{X,x} \to X$ (indeed, this is true for any local ring). So we need $\mathscr{O}_{X,x} \to A$ factors uniquely through $\mathscr{O}_{X,x} \to \hat{\mathscr{O}}_{X,x}$. But since A is an Artin ring, so some power $\mathfrak{m}_x^n$ maps to 0 in A. IE $\mathscr{O}_{X,x} \to A$ factors through some $\mathscr{O}_{X,x}/\mathfrak{m}_x^n$. So we get a factorization through $\hat{\mathscr{O}}_{X,x}$.

The second part is that $\hat{\mathscr{O}}_{X,x}/\mathfrak{m}_x^n$ and $R/\mathfrak{m}_R^n$ both give Artin rings for all n. Using a Yoneda style trick, construct compatible maps $R \to \hat{\mathscr{O}}_{X,x}/\mathfrak{m}_x^n$ and $\hat{\mathscr{O}}_{X,x} \to R/\mathfrak{m}_R^n$, we construct ???? □

Remarks: What data is in $\hat{\mathscr{O}}_{X,x}$?

1. Dimension of X at x
2. Singularity type of X at x, something similar to a local ring of an analytic space
3. eg: Cohen then X is smooth over k of dimension n we have $\hat{\mathscr{O}}_{X,x} \simeq k[[x_1, \ldots, x_n]]$.
4. eg: $y^2 = t^3 - t^2$ is a nodal cubic, then $\hat{\mathscr{O}}_{X,x} \simeq k[[u, v]]/(uv)$.
5. eg: $y^2 = t^3$ is a cuspidal cubic, get $k[[x, y]]/(y^2 - x^3) \not\simeq k[[s]]$ even though homeomorphic.

The functors of interest.

We work in the relative setting. We'll fix Λ a complete local Noetherian ring with residue field k, we'll consider $Art(\Lambda, k)$ of Artin local Λ-algebras with residue field k.

Nonstandard terminology: A predeformation functor is a functor $F : Art(\Lambda, k) \to Set$ such that $F(k)$ is the one-point set.

Roughly, these arise by considering families over $\operatorname{Spec} A$ restricting to a fixed object over $\operatorname{Spec} k$. Starting with a global moduli functor, can obtain a predeformation functor by choosing an object over k and restricting attention to Artin rings (not necessarily a good idea).