30 NOVEMBER 2000.

Abstract. The Hilbert scheme of closed and open subschemes of the line.

§1. Introduction.

The purpose of the note is to give concrete examples of Hilbert schemes; to show how the schemes satisfy their defining universal properties, and to show that things that can go wrong when only considering the set of rational points of the Hilbert functor. The definition of the Hilbert functor of n-points on a k-scheme W is as follows.

Definition. The Hilbert functor of n-points on W is the functor from the category of noetherian k-schemes to sets, determined by sending a scheme T to the set $\mathcal{H}^n_X(T)$ of closed subschemes $Z \subseteq W \times_k T$, such that the projection map $Z \to T$ is flat, finite and of rank n.

The three examples that I will describe are the following

(1) The Hilbert scheme of the line. Let $W = \mathbf{A}^1$, the affine line. It is clear that the set of 2-points on the affine line is in bijection with the group quotient $\mathbf{A}^1 \times \mathbf{A}^1/\Sigma_2$. In Section 2 we will see that $\mathbf{A}^1 \times \mathbf{A}^1/\Sigma_2$ is not only a good candidate for the Hilbert scheme of two points on W, it acutally is the Hilbert scheme.

(2) The Hilbert scheme of a thick point on the line. Let $W = \operatorname{Spec}(k[x]/(x^m))$, a thick point on the line. Since $W \subseteq \mathbf{A}^1$ is a closed subscheme of the line, the Hilbert scheme of two points on W is a closed subscheme of $\mathbf{A}^2$, the Hilbert scheme of two points on the line. It is clear that the only closed subscheme of W of length 2 is $\operatorname{Spec}(k[x]/(x^2)$, provided that $m \geq 2$ which we will assume. If $m < 2$ then the Hilbert scheme is the empty set.

    We have that the Hilbert scheme of two points on W only has one closed point, and consequently the Hilbert scheme of 2-points on W is affine and given the spectrum of a local artin ring.

    Which artin ring, what is dimension of the k-vector space of the artin ring, and an explanation for that particular dimension.

(3) The Hilbert scheme of the local ring of a point on the line. Let $W = \operatorname{Spec}(k[X]_{(X)})$, the spectrum of the local ring of the origin at the line. Again we have that the only closed subscheme of W of length 2 is $\operatorname{Spec}(k[X]/(X^2))$, and consequently the Hilbert scheme $\operatorname{Hilb}^2_W = \mathbf{H}$ has only one k-rational point. However, the Hilbert scheme is not given by some artinian ring. It turns out that the Hilbert scheme is an intergral scheme of dimension 2, but not a variety.

    The crucial fact is that $W = \operatorname{Spec}(k[X]_{(X)})$ is not a subscheme of the line, and consequently the Hilbert functor/scheme of W is not a subfunctor/subscheme of the Hilbert functor/scheme of the line $\mathbf{A}^1$.

§2. Hilbert scheme of the line.

2.0. Notation. Throughout the note we will let X, t and u be algebraic independent variables over the base field k. The group of 2-letters Σ_2 acts on the polynomial ring $k[t,u]$ by permuting the variables. The invariant ring $\Lambda_2 = k[t+u, tu]$ is again a polynomial ring in the variables $t+u$ and tu. In the polynomial ring $\Lambda_2[X]$ we have the element

$$\Delta_2(X) = X^2 - (t+u)X + tu = (X-t)(X-u).$$

Proposition 2.1. *The Hilbert functor of two points on* $W = \mathbf{A}^1$ *is represented by the affine scheme*

$$\mathbf{H} = \operatorname{Spec}(k[t+u, tu]).$$

The universal family is given by

$$V(X^2 - (t+u)X + tu) \subseteq \mathbf{H} \times W.$$

Proof. Since $X^2 - (t+u)X + tu$ is a monic polynomial of degree 2, we have that that $V(X^2 - (t+u)X + tu)$ is an element of $\mathcal{H}^2_W(\mathbf{H})$. Hence there is an induced morphism of functors

$$\operatorname{Hom}(-, \mathbf{H}) \to \mathcal{H}^2_W, \qquad (2.1.1)$$

which we claim is an isomorphism. It is enough to check that (2.1.1) is an isomorphism for affine schemes.

Let Z be an $\operatorname{Spec}(A)$-valued point of the Hilbert functor, where A is some k-algebra. We have that Z is affine and given by an ideal $I \subseteq A \otimes_k k[X] = A[X]$. By possibly shrinking $\operatorname{Spec}(A)$ we may assume that the quotient $A[X]/I$ is free of rank 2 as an A-module. We have the following surjective map

$$A[X] \to A[X]/I = A \oplus A, \qquad (2.1.2)$$

which has to send the variable X to an endomorphism of $A \oplus A$. Let $F(X) \in A[X]$ be its characteristic polynomial. Then $F(X) = X^2 - a_1X + a_2$, for some elements $a_1, a_2 \in A$. By the Cayley Hamilton Theorem $F(X)$ is in the kernel of the map (2.1.2). Thus the induced map

$$A[X]/(F(X)) \to A[X]/I$$

is a surjective map between two free modules of the same rank, thence an isomorphism. Clearly $F(X)$ is the unique monic polynomial that generate I. It follows that the k-algebra homomorphism $u_F : k[t+u, tu] \to A$, determined by sending $(t+u) \mapsto a_1$ and $tu \mapsto a_2$, is the unique morphism $\operatorname{Spec}(A) \to \mathbf{H}$ such that

$$\operatorname{Spec}(A) \times_{\mathbf{H}} V(X^2 - (t+u)X + tu) = Z.$$

Thus $\Phi(\operatorname{Spec}(A)) : \operatorname{Hom}(\operatorname{Spec}(A), \mathbf{H}) \to \mathcal{H}^2_W(\operatorname{Spec}(A))$ is a bijection for all k-algebras A. We have shown that Φ is an isomorphism of functors.

Remark. We of course knew this as it has been pointed out earlier by R. Vakil in the course. Now it may seem that we have chosen an odd presentation for the affine 2-space as $\operatorname{Spec}(k[t+u, tu])$, we could just as well have taken $\mathbf{H} = \operatorname{Spec}(k[t,u])$. The reason for why the given presentation is better, or even the right one, might be clear in the next two sections.

§3. Hilbert scheme of a thick point on the line.

To understand which artin ring that represents the Hilbert functor of 2-points on $W = \mathrm{Spec}(k[X]/(X^m))$, we have to include the following.

3.0. A Canonical map. Let $k[x_1, x_2, \dots, x_m]$ be the polynomial ring in m-variables over k. Let $\Lambda_m = k[\sigma_1, \dots, \sigma_m]$ be the subring of symmetric tensors, which is a polynomial ring in the elementary symmetric functions $\sigma_1, \dots, \sigma_m$, defined by

$$\Delta_m(X) = \prod_{i=1}^{m}(X - x_i) = X^m - \sigma_1 X^{m-1} + \dots + (-1)^m \sigma_m$$

in the polynomial ring $\Lambda_m[X]$. In the polynomial ring $k[x_1, \dots, x_m][X]$ we split our product as

$$\begin{aligned}\prod_{i=1}^{m}(X - x_i) &= \prod_{i=1}^{m-2}(X - x_i) \prod_{j=m-1}^{m}(X - x_j) \\ &= \Big(X^{m-2} - s_1 X^{m-3} + \dots + (-1)^{m-2} s_{m-2}\Big)\Big(X^2 - (t+u)X + (tu)\Big),\end{aligned}$$

where $s_1, \dots, s_{m-2}$ are the elementary symmetric functions in $x_1, \dots, x_{m-2}$, and where we set $t = x_{m-1}$ and $u = x_m$. In the polynomial ring $k[x_1, \dots, x_{m-2}, t, u]$ we get the following relations

$$\sigma_i = s_i + s_{i-1}(t+u) + s_{i-2}tu,$$

for each $i = 1, \dots, m$, where we use the conventions that $s_0 = 1$ and that $s_k = 0$ if $k \geq m-2$ or if $k < 0$. Consequently we have a natural inclusion of rings

$$\Lambda_m \subseteq \Lambda_{m-2} \otimes_k k[t+u, tu] = \Lambda_{m-2} \otimes_k \Lambda_2,$$

which makes $\Lambda_{m-2} \otimes_k \Lambda_2$ into a Λ_m-module.

Lemma 3.1. *$\Lambda_{m-2} \otimes_k \Lambda_2$ is free Λ_m-module of rank $\binom{m}{2}$.*

Proof. We know (Exercise (?) in [1]) that the polynomial ring $V = k[x_1, \dots, x_m]$ is a free $\Lambda_{m-n} \otimes_k \Lambda_n$-module of rank $(m-n)!n!$, for any non-negative integer $n \leq m$. In particular $k[x_1, \dots, x_m]$ is free of rank $m!$ as an Λ_m-module, and free of rank $(m-2)!2!$ over $\Lambda_{m-2} \otimes_k \Lambda_2$. From the same exercise we get that it is possible to construct a basis B for V over Λ_m, where a subset of B is a basis for V over $\Lambda_{m-2} \otimes_k \Lambda_2$, from which it follows that $\Lambda_{m-2} \otimes_k \Lambda_2$ is a free Λ_m-module of rank $\frac{m!}{(m-2)!2!} = \binom{m}{2}$.

In view of Proposition (2.1), or its generalization, we thus have the following canonical morphism of schemes

$$\pi : \mathrm{Hilb}^{m-2}_{\mathbf{A}^1} \times_k \mathrm{Hilb}^{2}_{\mathbf{A}^1} \to \mathrm{Hilb}^{m}_{\mathbf{A}^1} .$$

Proposition 3.2. *Let $W = \mathrm{Spec}(k[X]/(X^m))$. Let $\pi : \mathrm{Spec}(k) \to \mathrm{Hilb}^m_{\mathbf{A}^1}$ be the point corresponding to the k-algebra homomorphism $k[\sigma_1, \dots, \sigma_m] \to k$ sending σ_i to zero, for all $i = 1, \dots, m$. Then we have the following facts*

1. *The Hilbert scheme Hilb^2_W of 2-points on W is $\pi^{-1}(\mathrm{Spec}(k))$.*
2. *$\mathrm{Hilb}^2_W = \mathrm{Hilb}^{m-2}_W$.*
3. *The vector space dimension of $\Gamma(\mathrm{Hilb}^2_W)$ is $\binom{m}{2}$.*

Proof. Let H be the affine coordinate ring of the fiber $\pi^{-1}(\mathrm{Spec}(k))$. Then H is a quotient of $\Lambda_{m-2} \otimes_k \Lambda_2$ and we let $V = H[X]/(X^2 - (t+u)X + tu)$ where $(t+u)$ and tu are the residue classes of $t+u$ and tu. Clearly V is a free H-module of rank 2. Morover, from the splitting of $\prod_{i=1}^m (X - x_i)$ above and the definition of H we have that

$$X^m = \Delta_{m-2}(X)(X^2 - (t+u)X + tu) \in H[X].$$

That is $(X^m) \subseteq (X^2 - (t+u)X + tu)$ in $H[X]$, and consequently we have an induced morphims of functors

$$\Phi : \mathrm{Hom}(-, \mathrm{Spec}(H)) \to \mathcal{H}^2_W. \tag{3.2.1}$$

We shall show that Φ is an isomorphism. Let A be an k-algebra and let $Z \in \mathcal{H}^2_W(\mathrm{Spec}(A))$ be an A-valued point. Then, by possibly shrinking $\mathrm{Spec}(A)$, we have that Z is given by an ideal I such that $A[X]/I$ is a free A-module of rank 2. We have the following sequence

$$A[X] \to A[X]/(X^m) = A \otimes_k k[X]/(X^m) \to A[X]/I.$$

Let J be the kernel of the composite map. Then by Proposition (2.1) we have that J is generated by a unique monic polynomial $F(X) = X^2 - a_1X + a_2 \in A[X]$. Since the composite map factors through $A[X]/(X^m)$ we have that $F(X) \in (X^m)$. That is there exist a $G(X) \in A[X]$ such that

$$X^m = F(X)G(X). \tag{3.2.2}$$

It follows that $G(X)$ is a monic polynomial of degree $m-2$. It follows readily from the fact that $A[X]$ is a free A-module with basis $\{X^i\}_{i \geq 0}$ that $G(X)$ is uniquely determined by $F(X)$, hence by I. The coefficients of $F(X)$ and $G(X)$, determines an A-algebra homomorphism

$$u_I = u_G \otimes u_F : \Lambda_{m-2} \otimes_k \Lambda_2 \to A. \tag{3.2.3}$$

What we need to show is that u_I factors through H. The ring H is the quotient of $\Lambda_{m-2} \otimes_k \Lambda_2/K$, where K is the extension of the ideal $(\sigma_1, \dots, s_m) \subseteq \Lambda_m$. It follows that K is generated by $f_1, \dots, f_m$, where

$$f_i = s_i + s_{i-1}(t+u) + s_{i-2}tu.$$

To show that the map (3.2.3) factors through H we need to show that $u_I(f_i) = 0$ for all $i = 1, \dots, m$. By expanding the product (3.2.2), and using the definition of the maps u_F and u_G, we get that

$$X^m = F(X)G(X) = X^m - u_I(f_1)X^{m-1} + u_I(f_2)X^{m-2} + \cdots + (-1)^m u_I(f_m)$$

in $A[X]$. The A-module $A[X]$ has a basis $\{X^i\}_{i\geq 0}$, which combined with the identity above implies that $u_I(f_i)=0$ for $i=1,\dots,m$. This proves Assertion (1).

It is clear that Assertion (1) and symmetri implies Assertion (2), and that Assertion (1) and Lemma (3.1) implies Assertion (3).

Remark. By looking at the defining equations $f_1,\dots,f_m$ one realizes that the Hilbert scheme of 2-points on $W=\operatorname{Spec}(k[X]/(X^m))$ can be realized as a closed subscheme of the affine plane, cut out by two equations. Simply because the variable s_i occurs as a linear expression in the function f_i, for $i=1,\dots,m-2$. It is however hard to give the defining equations explicitely.

Example 3.3. How does the Hilbert scheme look like, when embedded into the plane. As an example, let $m=5$. We then have the following 5 equations

$$s_1+x,\ s_2+s_1x+y,\ s_3+s_2x+s_1y,\ s_3x+s_2y,\ s_4y.$$

By making the substiutions $s_1=-x$, $s_2=-s_1x-y=x^2-y$ and $s_3=2xy-x^3$, we get that the Hilbert scheme of 2-points on $\operatorname{Spec}(k[x]/(x^5))$ is the closed subscheme of the plane given as

$$\operatorname{Spec}(k[x,y]/(3x^2y-x^4-y^2,2xy^2-x^3y)).$$

Remark. We chose a special section $\operatorname{Spec}(k)\to \operatorname{Hilb}^m_{\mathbf{A}^1}$ in our Proposition. Clearly, there is an analogous statement for any point $\operatorname{Spec}(k)\to \operatorname{Hilb}^m_{\mathbf{A}^1}$. To explain the dimension formula of Proposition (3.2), one should instead of the thick point $\operatorname{Spec}(k[X]/(X^m))$ consider m-different reduced points on the line. There are clearly $\binom{m}{2}$ different ways to pick 2 points out of m points.

§4. Hilbert scheme of the local ring of the line.

Let A be an k-algebra. The tensorproduct $A\otimes_k k[X]_{(X)}$ is then the fraction ring of $A[X]$ with respect to the multiplicatively closed set

$$T=\{f(X)\in A[X]\mid f(X)\in k[x]\subseteq A[X], f(0)\neq 0\}.$$

Lemma 4.1. *If $I\subseteq A\otimes_k k[X]_{(X)}$ is an ideal such that the quotient ring is a free A-module of rank n, then the ideal is generated by a unique monic polynomial $F(X)\in A[X]$, of degree n. That is, the fraction map*

$$A[X]/(F(X))\to A\otimes_k k[X]_{(X)}/(F(X))$$

is an isomorphism.

Proof. This is a non-trivial fact, but somehow expected. We do only want (families) of points in $\operatorname{Spec}(k[X]_{(X)})$ that actually come from a family of points on the line. For a direct proof of this see [2].

Let $V=\Lambda_m[X]/(\Delta_m(X))$, where $\Delta_m(X)$ is the symmetric expansion of the product $\prod_{i=1}^m(X-x_i)$. We have that V is a free Λ_m-module of rank m

Lemma 4.1. *Let $f \in \Lambda_m[X]$. The Λ_2-linear endomorphism on $\Lambda_2[X]/(\Delta_2(X))$ sending $a \mapsto a\bar{f}$, where $\bar{f}$ is the residue class of f modulo $(\Delta_2(X))$, has characteristic polynomial*

$$X^2 - (f(t) + f(u))X + f(t)f(u).$$

Proof. Let N be a square matrix with coefficients in a ring R. If the characteristic polynomial of N splits into linear factors over R, that is $\det(X - N) = \prod(X - \alpha_i)$ in $R[X]$, then for any $f(X) \in R[X]$ the matrix $f(N)$ has characteristic polynomial $\prod(X - f(\alpha_i))$. A proof of this fact for general rings R is found in [3].

One compute by hand that the endomorphism $a \mapsto ax$ on $\Lambda_2[X]/(\Delta_2(X))$ has characteristic polynomial $\Delta_2(X) = (X-t)(X-u)$. It follows that the characteristic polynomial of $a \mapsto af$ is

$$(X - f(t)(X - f(u)) = X^2 - (f(t) + f(u))X + f(t)f(u).$$

Proposition 4.4. *The Hilbert scheme of 2 points on $W = \mathrm{Spec}(k[X]_{(X)})$ is affine and given as*

$$\mathbf{H} = \mathrm{Spec}(k[t + u, tu]_U),$$

where U is the multiplicatively closed subset

$$U = \{f(t)f(u) \mid f(X) \in k[X], where f(0) \neq 0\}.$$

Proof. From Lemma (4.1) we have that an A-valued point of the Hilbert functor of two points on $k[X]_{(X)}$ is uniquely given by a monic polynomial $F(X) \in A[X]$ such that the fraction map

$$A[X]/(F(X)) \to A \otimes_k k[X]_{(X)}/(F(X)) \tag{4.4.1}$$

is an isomorphism. Let $F(X) = X^2 - a_1X + a_2 \in A[X]$, and let $u_F : \Lambda_2 \to A$ be the k-algebra homomorphism sending $t + u \mapsto a_1$ and $tu \mapsto a_2$.

The fraction map (4.4.1) is an isomorphism if and only if for all $f(X) \in k[X] \setminus (X)$, the residue class of $f(X)$ modulo the ideal $(F) \subseteq A[X]$ is a unit. The class of $f(X) \in A[X]/(F)$ is a unit if and only if the endmorphism $a \mapsto af$ is a unit, which again is equivalent with $\det(a \mapsto af) \in A$ is invertible. By Lemma (4.2) it follows that

$$\det(a \mapsto af) = u_F(f(t)f(u)).$$

So, if (4.3.1) is an isomorphism the the morphism $u_F : \Lambda_2 \to A$ factors via the fraction map $H = k[t + u, tu]_U$. And conversely; a k-algbera homomorphism $H \to A$, gives by composition a map $u'_F : \Lambda_2 \to A$. And since $u'_F(f(t)f(u))$ is then mapped to a unit in A for all $f(X) \in k[X] \setminus (X)$ we have that (4.3.1) is an isomorphism.

Example. It is instructive to draw a picture of the $\mathrm{Spec}(k[t+u, tu]_U)$. To describe the Hilbert scheme we must controll which prime ideals P in $A = k[t + u, tu]$ that meet U and which do not. For simplicity we assume that the base field k is algebraically closed.

If $f(X) = a + X$, with $a \neq 0$, then the element

$$f(t)f(u) = a^2 + 2a(u+t) + ut$$

is in the multiplicatively closed set $U \subseteq k[t+u, tu]$.

Let P be a maximal ideal different from the origin, given by the coordinates (α, β). It is clear that we can find a nonzero $a \in k$ that solves the quadratic equation

$$a^2 + 2a\alpha + \beta = 0,$$

which means that $P \cap U \neq \emptyset$. Consequently all k-rational points, that is maximal ideals, different from $(0,0)$ will meet U, and not correspond to points in the scheme $\mathrm{Spec}(k[t+u, tu]_U)$. On the other hand it is easy to see that $(t+u, tu) \cap U = \emptyset$ since all elements in U have a constant term different from zero.

Thus $(0,0)$ is the only k-rational point of the Hilbert scheme, but there are other closed points. Let $P \subseteq k[t+u, tu]$ be the prime ideal generated by an irreducible polynomial $P(t+u, tu)$, where $P(t+u, tu)$ has a constant term different from zero, and where $P(t+u, tu)$ is also irreducible in $k[t, u]$. (Take for instance $P(t+u, tu) = 1 + t + u$. Since $P(t+u, tu)$ has a non-zero constant term the prime ideal P is not included in $(t+u, tu)$. And since $P(t+u, tu)$ is irreducible in $k[t, u]$, it follows that $P \cap U = \emptyset$. Because, otherwise $P(t+u, tu)g(t, u) = f(u)f(t)$, for some $g(t, u) \in k[t, u]$. And since $P(t+u, tu)$ is irreducible it has to be a prime factor of either $f(u)$ or $f(t)$. In anyway a polyomial in one variabel, which then cannot be symmetric.

So, since $(P(t+u, tu))$ does not meet U, and is not included in $(t+u, tu)$ it is maximal in the fraction ring $k[t+u, tu]_U$. Any prime ideal $M \subseteq k[t+u, tu]$ properly containing $(P(t+u, tu))$ is a maximal ideal different from $(t+u, tu)$, and becomes invertible in the fraction ring.

Since the maximal ideal $(t+u, tu) \subset k[t+u, tu]$ extends to a prime ideal in the fraction ring $k[t+u, tu]_U$, it follows that the Hilbert scheme $\mathrm{Spec}(k[t+u, tu]_U)$ has dimension two. The dimension at the other closed points is however 1.

Remark. We have chosen a particular fraction ring $k[X]_{(X)}$ in the last section. It is clear that the proof of the Proposition works for any fraction ring $k[X]_S$, with the necessary modifications of course.

References

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