MODULI SPACES AND DEFORMATION THEORY, CLASS 7

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1. Left-over remarks

Reason for mentioning adding a final object $\mathcal{M}$ to moduli topology: This is a straightforward way of making sense of etale cohomology of moduli space. All we need are (i) tensor products (done via isom), and (ii) definition of etale coverings. Etale, and all curve over algebraically closed fields exist.

Jason has pointed out another good reference for stacks: Andrew Kresch's homepage at penn. http://www.math.upenn.edu/˜kresch/teaching/stacks.html

Other remarks: The Hilbert functor $Hilb(X/S)$ is the functor which associates to a morphism $f : T \to S$, the set of all closed subschemes of $T \times_S X$ which are flat *and proper* over T. Without the "proper" condition, $Hilb(X/S)$ isn't necessarily representable after all. Assuming this, it is true that $Hilb(X/S)$ is representable by a separated S-scheme which is a disjoint union of quasi-projective S-schemes, if X is also finitely presented over S. (Note that quasiprojective doesn't imply finitely presented: let S be some hideous pathological scheme, and X a closed subscheme given by a non-finitely generated ideal.)

2. Stacks

Topologies on X: classical (if complex), Zariski, etale, smooth, flat.

Important exercise. If X is a scheme, then consider the following functor: $Y \to \mathrm{Mor}(Y, X)$. Of course you've seen this functor before. Show that this is a

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sheaf in the Zariski topology. State precisely what results would be necessary for it to be true in the etale topology.

We want to generalize this notion to "groupoids over {Sch}". We know longer have a functor to sheaves, or groupoids. So here's the right stand-in.

Definition. A groupoid F over {Sch} is a *stack* if

(i) (Isomorphisms are a sheaf) For any $X \in \{\text{Sch}\}$ and any two objects a and b in the groupoid $F(X)$, the functor $Isom_X(a, b) : \{\text{Sch}\} \to$ (Sets) (which associates to morphism $f : Y \to X$ the set of isomorpihsms in $F(Y)$ between f^*a and f^*b, is a sheaf in the etale topology.

Exercise. State precisely what you need for this to be true for a moduli groupoid.

(ii) Let $\{X_i \to X\}$ be a covering of $X \in \{\text{Sch}\}$ in the etale topology. Let $a_i \in F(X_i)$, and let

$$\phi_{ij} : a_j|_{X_i \times_X X_j} \to a_i|_{X_i \times X_j}$$

be isomorphisms in $F(X_i \times_X X_j)$ satisfy the cocycle condition. Then there is $a \in F(X)$ with isomorphisms $\psi_i : a|_{X_i} \tilde{\to} a_i$, such that

$$\psi_{ij} = (\psi_i|_{X_i \times_X X_j}) \circ (\psi_j|_{X_i \times_X X_j})^{-1}.$$

Translation: gluability for objects; Identify for objects follows from the first one. Hence objects "form a sheaf". (Some funkiness here too — explain why we can't hope for anything more.)

Exercise. Check precisely what you need for this to be true for a moduli groupoid. Show that these conditions are immediate if we were working in the Zariski topology.

In general, you can have stacks in various topologies.

Definition. A stack F is of Deligne-Mumford type (i.e. is a DM-stack) if

(i) The diagonal Δ_F is representable, quasicompact, and separated.
(ii) There is a scheme U and an etale surjective morphism $U \to F$. Such a morphism $U \to F$ is called an *atlas*.

(Remark: I'd prefer if you didn't call it an algebraic stack.)

Checking this isn't so bad for a moduli functor. We already have the representability of the diagonal in "good situations". More generally, this comes down to checking that the Isom sheaf is representable, separated, and quasicompact, and then finding an atlas.

Exercise: Say this precisely.

Remark: forces finite automorphism group of objects. Precisely: ...

You can make sense of the dimension of a DM-stack as well. (Explain.)

2.1. **Random remarks.** Jason's example of elliptic curves:

Even in the case when $S = Spec(k)$ and X/S is projective, it can happen that $Aut(X/S)$ is not quasi-compact. An example of a projective X with $Aut(X/S)$ not quasi-compact is $X = E \times E$ with E an elliptic curve over S. For any matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in $SL(2, Z)$, we can form the automorphism $f : E \times E \to E \times E$ by $f(x, y) = (ax + by, cx + dy)$. Thus $Aut(X/S)$ contains a copy of $SL(2, Z)$, so it isn't quasi-compact.

Define Artin stack. Additional condition: lfpr; can define dimensions, so $\dim BG$ is -3.

2.2. **Summary of Deligne-Mumford stacks.** How we got there: contravariant functor from sheaves to sets. Then to groupoids. Then a moduli groupoid (or a groupoid over $\{Sch\}$).

Stack: isomorphisms form a sheaf; objects "form a sheaf".

Think of it as a scheme, with stacky points.

The great thing is that you can work on your atlas U, as everything you're interested in descends.

2.3. **Sheaves on Deligne-Mumford stacks.** Define Quasicoherent sheaves, coherent sheaves, locally free sheaves (vector bundles), homomorphisms of locally free sheaves.

A *quasicoherent sheaf* S on a Deligne-Mumford stack F is the following data (i) to each atlas $U \to F$ a quasicoherent sheaf S_U on U. (Yes, a Zariski-sheaf!) (ii) For each commutative diagram $U \xrightarrow{\phi} V$ over F with U and V atlases, an isomorphism $a_\phi : S_U \to \phi^* S_V$. These isomorphisms satisfy the cocycle condition, i.e. for any three atlases $U \xrightarrow{\phi} V \xrightarrow{\psi} W$ over F, we have

$$a_{\psi \circ \phi} = a_\phi \circ \phi^* a_\psi : S_U \xrightarrow{\sim} (\psi \circ \phi)^* S_W = \phi^*(\psi^* S_W).$$

A coherent sheaf and locally free sheaf are defined similarly. A homomorphism is these things is what you think it is. You can define inverse and direct images.

Exercise: This is sensible because all 4 concepts descend under etale covers.

Example: Hodge bundle. L_i, tautological line bundles. Structure sheaf.