MODULI SPACES AND DEFORMATION THEORY, CLASS 2

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Contents

1. Schemes as functors; Yoneda's lemma 1
2. Representable functors 2
2.1. Properties of functors (esp. representable functors) and morphisms about functors (representable morphisms) 3
3. Moduli functors 4
4. Example 1 in detail 6

Topics today:

1. Yoneda's lemma
2. Representable functors
3. Moduli functors and fine moduli spaces

The theme will be: schemes are functors, but not all functors are schemes. Some functors aren't so bad even if they aren't schemes. We'll see examples.

Administrative stuff: if you haven't signed up on this sheet, please do so after class. I'm still expecting to put each classes' notes up, usually a few days after the class. I'll e-mail you all when Classes 1 and 2 go up. Ana-Maria Castravet (noni@math) will be the (half-)grader for this course. I'll be giving out problems to hand in from time to time. The "exercises" I mention aren't for handing in (although they may become "problems" later), but they are for thinking about and talking about.

Also, whenever I mention the category of schemes, I will likely mean the category of schemes of finite type of $\mathbb{Z}$.

1. Schemes as functors; Yoneda's lemma

In this discussion, the category of schemes can be replaced by *any* category.

Date: Tuesday, September 12, 2000.

Contravariant functor h_X from schemes to sets, $Y \mapsto \mathrm{Mor}(Y, X)$. $Y \to Z$ gives $\mathrm{Mor}(Z, X) \to \mathrm{Mor}(Z, X)$. This is called the *functor of points.*

Here's why. If X is a scheme over $\mathbb{C}$, then the image of $\mathbb{C}$ are the complex points of X. More generally, morphisms from a scheme T to a scheme X are called *T-valued points of X.*

Key observation. $Y \to X$, $\mathrm{Mor}(X, X) \to \mathrm{Mor}(Y, X)$, *identify maps to the morphism.*

1.1. *Yoneda's Lemma.* — h_X and $h_{X'}$ are the identified functors. Then X and X' are canonically isomorphic.

From the identification of $\mathrm{Mor}(X, X)$ with $\mathrm{Mor}(X, X')$, and the identification of $\mathrm{Mor}(X', X')$ with $\mathrm{Mor}(X', X)$, we get natural morphisms $m_{XX'} : X \to X'$ and $m_{X'X} : X' \to X$. We just need to check that $X \to X' \to X$ is the identity.

Look at

$$\begin{array}{ccccc}
X & \stackrel{m_{XX'}}{\to} & X' & \stackrel{m_{X'X}}{\to} & X \\
 & & & & \\
? & & m_{XX'} & & id \\
\mathrm{Mor}(X, X) & \leftarrow & \mathrm{Mor}(X', X) & \leftarrow & \mathrm{Mor}(X, X) \\
|| & & || & & || \\
\mathrm{Mor}(X, X') & \leftarrow & \mathrm{Mor}(X', X') & \leftarrow & \mathrm{Mor}(X, X') \\
m_{XX'} & & id & & m_{XX'}
\end{array}$$

If $? = id$, we're done, and it is. $\square$

2. Representable functors

A functor h is *representable* (in the category of schemes) if there is a scheme X and an isomorphism of functors $h \to h_X$.

Examples of representable functors.

You may recall, in some subject or other, constructions of the following sort. (Think: tensor product, normalization, etc. in algebraic geometry and commutative algebra. But also direct limits, indirect limits, and many other things.) The properties of the construction are described (which can usually be translated into functorial language). The book then says "by the usual arguments, should this thing exist, it is defined up to unique isomorphism". Finally, the book goes about constructing the object.

Translation/example: you're working in some category, e.g., schemes. The book defines a functor, e.g. tensor product. Say you have two schemes X and Y, and morphisms $f : X \to Z$, $g : Y \to Z$. Then there is a contravariant functor taking a scheme W to the set

$$\{(f', g') : f' : W \to Y, g' : W \to Z, \text{ and } g \circ f' = f' \circ g\}.$$

(Write this in diagram form.) This is an honest-to-goodness functor; for example, given a morphism $W \to W'$, we have a morphism of these sets.

Then by Yoneda's lemma, if this functor is isomorphic to h_V and $h_{V'}$, then V and V' are canonically isomorphic. Hence we can call V the *tensor product* of X and Y over Z (with f and g implicit), and we've shown that it is unique (up to unique isomorphism).

Then we go about the business of constructing *a* tensor product, proving that *the tensor product is representable in the category of schemes.*

(In this way, can also define $\underline{\text{Spec}}$ of a quasicoherent sheaf of algebras, and $\mathbb{P}$, and *Red*, and $Norm$. Also direct limits and indrect limits of nice categories.)

2.1. **Properties of functors (esp. representable functors) and morphisms about functors (representable morphisms).** You can recover many things about schemes by looking at their corresponding functors.

Suppose you have a scheme X. Question: what are the $\mathbb{C}$-valued points of X, in terms of h_X?

(I'll use this language a lot; please stop me when it gets confusing.)

A big theme of the first couple of weeks of this course is that functors aren't terribly worse than schemes, and some are only a very little worse. For example, what are the $\mathbb{C}$-valued points of a functor F? Answer: $F(\operatorname{Spec}\mathbb{C})$.

I'll now define a proper morphism of functors. Suppose you have a morphism of schemes $X \to Y$, that is proper. Then for any map $Z \to Y$, $Z \times_Y X \to Y$ is proper too. If this is true for all $Z \to Y$, then $X \to Y$ was proper to begin with. (That's true for stupid reasons; just take $Z = Y$ and $Z \to Y$ the identity.)

Let me recast this in funtorial language. We have a morphism of representable functors $h_Y \to h_X$ (recall that these functors are contravariant!). For any morphism of representable functors $h_Y \to h_Z$, there is a new functor $h_{Z \times_Y X}$ that is actually representable, so we get the following diagram. (Draw same diagram with arrows reversed.)

Note to myself: Why is $h_Y \to h_Z$ representable?!

So now let me define a proper morphism of functors. First, I'll tell you what a *representable morphism of functors* is.

Suppose you have a morphism of functors $G \to F$, such that for any morphism $G \to h_Z$, the fiber-product functor is also representable, say h_W. (Draw diagram.)

Definition. Such a morphism of functors is called a *representable morphism.*

Then suppose for all such $G \to h_Z$, the induced morphism $W \to Z$ is proper. Then we say $G \to F$ is proper.

This sort of definition works well with any sort of property of schemes that is preserved by base change. For example, smooth, etale, flat, open immersion, closed immersion.

3. Moduli functors

I'll now give you examples of moduli functors, which are functors from schemes to sets of the following form:

$$W \mapsto \text{“Families over } W\text{”}.$$

Example 1: “Two points in $\mathbb{P}^1$*.* Precisely: to W, associate the sets of *Cartier divisors* on $\mathbb{P}^1 \times W$ of relative degree 2, not containing any fibers.

(Here the $\mathbb{P}^1$ is *fixed*.)

Note that this tells us what families are over *any* W, not just “nice” ones — for examples, we know what families are over any nonreduced scheme, such as $\operatorname{Spec} \mathbb{C}[\epsilon]/\epsilon^2$.

(Draw a picture.)

I'll come back to this example soon, in more detail.

The next functors are incredibly important.

Example 2: Families of curves. To each scheme S, we associate “families of nodal curves” over S. Precisely, a family of pointed nodal curves is a proper flat morphism $\pi : C \to S$ whose geometric fibers are connected, reduced and pure dimension 1, with at worst ordinary double points as singularities.

Geometric fiber: Base-change to an algebraically closed field.

Reality check question: what are the complex-valued points of this functor? Answer: complex nodal curves.

Important remark: flatness turns out to be the right concept; it gets rid of obnoxious things such as (fiber jumps) and (nonreduced) — very important remark, this tells us what families are over nonreduced bases.

Families of pointed curves: the above data, as well as n sections, labelled s_1, ..., s_n, that are disjoint, and contained in the smooth locus of π.

Families of stable pointed nodal curves. For any geometric fiber, any irreducible component of genus 0 has at least 3 special points; any irreducible component of genus 1 has at least 1 special points.

Sadly, none of these functors are representable (in the category of schemes), but the last one is very close.

Combinatorial exercise. Show that any stable curve has $2g - 2 + n > 0$.

Exercise. Show that this means no infintesimal autmorphisms as well.

You can refine these curve functors more; for example, you can require that the geometric fibers have arithmetic genus g. Then, in the stable case, you get the functor that is often called $\overline{\mathcal{M}}_{g,n}$. In the nodal case, you get the functor that is often called $\mathfrak{M}_{g,n}$.

Example 3: Families of closed subschemes (the Hilbert functor).

To each S, associate flat subschemes of $\mathbb{P}^n \times S$ over D.

You could further specify the Hilbert polynomial, as the Hilbert polynomial does not jump in families. More preicsely, one requires that the Hilbert polynomial is a given one for all geometric fibers.

This functor is representable, and this building block of Grothendieck's is the key to defining all sorts of other moduli spaces. More on this later.

3.1. Universal families.

Definition. If a moduli functor is representable by a scheme X, we say X is a fine moduli space for the moduli functor. Again, by Yoneda, a fine moduli space is defined up to unique isomorphism.

If a moduli functor is representable, then it has a universal family, which I'll define soon. I'll give the theoretical proof, but it will be clearer when we work through an example.

Suppose we have an isomorphism of functors, $F \xrightarrow{\sim} h_X$. Then we have a distinguished element of $h_X(X) = \mathrm{Mor}(X, X)$, the identity. This means that we have a distinguished element of $F(X)$, which we call the universal family. (Draw picture.) Moreover, because this is an isomorphism of functors, *any* family over *any* base is pulled back from it. (Say precisely.) This is basically a diagram-chase, best done in the privacy of your own home.

4. Example 1 in detail

To fix these ideas in your head, let me work through the first example in detail. Actually, for simplicity, let me use an easier functor. (It isn't hard to use this to understand the harder case. For more experienced people: as we do this, think of the harder case. Also, see why this is an open subfunctor of the previous functor.)

Recall that to S we associated Cartier divisors of relative degree 2 on $\mathbb{P}^1 \times S$, not containing any fibers. Let me instead look at Cartier divisors on $\mathbb{A}^1 \times S$, such that every geometric fiber is a length 2 subscheme. (Draw a picture, indicate that one point can't run off.)

4.1. *Claim. — This functor is represented by* $\mathbb{A}^2 = \operatorname{Spec} \mathbb{Z}[d,e]$.

Consider an affine open of $\operatorname{Spec} A$ of S. Then the family above it is a Cartier divisor on $\operatorname{Spec} A[x]$, given by something of the form $x^2 + bx + c$, where b and c are in A. This gives a morphism $\operatorname{Spec} A \to \operatorname{Spec} \mathbb{Z}[d,e]$ by $(d,e) \to (b,c)$. This morphism patches together. $\square$

We also see the universal family:

$$\begin{array}{rcl} (x^2+dx+e) = & X & \hookrightarrow \mathbb{A}^2_{d,e} \times \mathbb{A}^1_x \\ & \downarrow & \\ & \mathbb{A}^2_{d,e} & \end{array}$$

Hence this functor is representable.

The older functor, of length 2 subschemes of $\mathbb{P}^1$ rather than $\mathbb{A}^1$, is also representable, and here's the universal family.

$$\begin{array}{rcl} (cx^2+dxy+ey^2) = & X & \hookrightarrow \mathbb{P}^2_{c,d,e} \times \mathbb{P}^1_{x,y} \\ & \downarrow & \\ & \mathbb{P}^2_{c,d,e} & \end{array}$$

4.2. Remark. This is actually a Hilbert scheme!

That is the content of the following technical result. Don't worry about the details unless you want to, or unless you're an algebraic geometer.

4.3. *Theorem. —* Suppose S is a scheme. Then a Cartier divisor on $\mathbb{P}^1 \times S$ of relative degree 2 and containing no fibers is a closed subscheme of $\mathbb{P}^1 \times S$, flat over S, whose geometric fibers have length 2, and vice versa.

To prove this, I'm going to invoke two technical results that I'm not going to prove in class. The reason I'm telling them to you is so that you get a feel for what sorts of technical results are necessary.

One direction follows from:

4.4. *Claim.* — Suppose $\pi : Y \to X$ is a flat morphism, and D is a Cartier divisor on Y not containing any fibers of π. Then $D \to X$ is also flat.

Jason and I worked out the details a couple of days ago, and I'll later include the argument in the course notes. (Explain why it proves one direction of the result.) I'll also try to find a reference to EGA; apparently it's an important lemma of Grothendieck's.

The other direction follows from:

4.5. *Claim.* — Suppose $Y \to X$ is a smooth morphism, and $D \subset Y$ is a closed subscheme flat over X. Then D is a Cartier divisor on Y.

I think I see how to prove this, so again I'll later post the proof, and hopefully give a reference to EGA.

One moral: flatness was part of the definition of Examples 2 and 3, but it was secretly present in the definition of Example 1.